Find Taylor Series about $x_0 = 0$ for the following:

3. $\cos(x^2)$

   Feel free to use the result from [2].

   Since (on the interval of convergence), $\cos(x)$ actually **equals** $\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k)!}$, they are just two different ways of writing the same function.

   Thus

$$\cos(x^2) = \sum_{k=0}^{\infty}(-1)^k \frac{(x^2)^{2k}}{(2k)!} = \sum_{k=0}^{\infty}(-1)^k \frac{x^{4k}}{(2k)!} = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \dots.$$

Find Taylor Series about $x_0 = 0$ for the following:

4. $\int \cos(x^2)\ dx$

Then approximate $\int_0^1 \cos(x^2)\ dx$ accurate within 10^{-5}.

$$\begin{aligned}
\cos(x^2) &= \sum_{k=0}^{\infty}(-1)^k \frac{x^{4k}}{(2k)!} = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \dots \\
\Rightarrow \int \cos(x^2)\ dx &= \int 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \dots\ dx \\
&= x - \frac{x^5}{5 \cdot 2!} + \frac{x^9}{9 \cdot 4!} - \frac{x^{13}}{13 \cdot 6!} + \dots + C \\
&= C + \sum_{k=0}^{\infty}(-1)^k \frac{x^{4k+1}}{(4k+1) \cdot (2k)!}
\end{aligned}$$

4. (continued)

Approximate $\int_0^1 \cos(x^2)\ dx$ accurate within 10^{-5}.

$$\begin{aligned} \int \cos(x^2)\ dx &= C + \sum_{k=0}^{\infty} (-1)^k \frac{x^{4k+1}}{(4k+1)\cdot(2k)!} \\ \Rightarrow \int_0^1 \cos(x^2)\ dx &= \sum_{k=0}^{\infty} (-1)^k \frac{1}{(4k+1)\cdot(2k)!} - \sum_{k=0}^{\infty} 0 \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{1}{(4k+1)\cdot(2k)!} \end{aligned}$$

Alternating series, sequence of terms decreasing, so Alternating Series Test applies. $\left| \int_0^1 \cos(x^2)\ dx - S_N \right| \leq a_{N+1}$, so find N so

$$\frac{1}{\big(4(N+1)+1\big)\cdot\big(2(N+1)\big)!} \leq 10^{-5}.$$

4. Approximate $\int_0^1 \cos(x^2)\, dx$ accurate within 10^{-5}. (continued)

$\left|\int_0^1 \cos(x^2)\, dx - S_N\right| \leq a_{N+1}$, so find N so

$$\begin{aligned} \frac{1}{\big(4(N+1)+1\big)\cdot\big(2(N+1)\big)!} &\leq 10^{-5} \\ \frac{1}{(4N+5)\cdot(2N+2)!} &\leq 10^{-5} \end{aligned}$$

Through experimentation, I find that

$$\begin{aligned} \frac{1}{(4\cdot 2+5)\cdot(2\cdot 2+2)!} &= 0.00010684 \not\leq 10^{-5} \\ \frac{1}{(4\cdot 3+5)\cdot(2\cdot 3+2)!} &= 0.000001459 \leq 10^{-5} \end{aligned}$$

So $\displaystyle\int_0^1 \cos(x^2)\, dx = \sum_{k=0}^{3} (-1)^k \frac{1}{(4k+1)\cdot(2k)!} \pm 10^{-5}$.

1. Find a power series expansion of $\int_0^1 e^{-x^3}\,dx$. Approximate the value of this integral accurate within 0.001.

$$
\begin{aligned}
e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \quad &\Rightarrow \quad e^{-x^3} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{3k}}{k!} \\
&\Rightarrow \quad \int_0^1 e^{-x^3}\,dx = \int_0^1 \left(\sum_{k=0}^{\infty} \frac{(-1)^k x^{3k}}{k!} \right) dx \\
&\Rightarrow \quad \int_0^1 e^{-x^3}\,dx = \sum_0^{\infty} (-1)^k \left(\frac{x^{3k+1}}{(3k+1)k!} \right) \Bigg|_0^1 \\
&\Rightarrow \quad \int_0^1 e^{-x^3}\,dx = \sum_0^{\infty} (-1)^k \frac{1}{(3k+1)k!}
\end{aligned}
$$

1. (continued)

$$\int_0^1 e^{-x^3}\,dx = \sum_0^{\infty}(-1)^k \frac{1}{(3k+1)k!}$$

Approximating this within .001 is just like approximating any other alternating series – simply make $a_{k+1} < .001$.

$$\frac{1}{(3(k+1)+1)(k+1)!} < .001 \Rightarrow (3k+4)(k+1)! > 1000$$

Experimenting with Maple, I find that $N = 4$ will do.
Therefore, accurate to within .001,

$$\int_0^1 e^{-x^3}\,dx \approx 1 - 1/4 + 1/14 - 1/60 + 1/312 \approx .80797.$$

Compare this to the approximation Maple gives: .80751.

2. *A Power Series for π!*

2.1 Find a power series expansion for $\dfrac{1}{1+x^2}$.

Since a power series expansion for $\frac{1}{1-x}$ is

$$\frac{1}{1-x} = P(x) = 1 + x + x^2 + x^3 + \ldots,$$

we have

$$\frac{1}{1+x^2} = P(-x^2) = 1 - x^2 + x^4 - x^6 + \cdots$$

2.2 Find a power series expansion for $\arctan(x)$.

Therefore

$$\begin{aligned}\arctan(x) &= \int \frac{1}{1+x^2}\, dx = \int 1 - x^2 + x^4 - x^6 + \cdots \; dx \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\end{aligned}$$

2. *A Power Series for π!*

2.3 Find a power series expansion for $\frac{\pi}{4} = \arctan(1)$.

$$
\begin{aligned}
\arctan(x) &= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)} \\
\Rightarrow \frac{\pi}{4} &= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)}
\end{aligned}
$$

2.4 Find a power series expansion for π.

$$
\begin{aligned}
\pi &= 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right) \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k \cdot 4}{(2k+1)}
\end{aligned}
$$