Let $P(x)$ be the power series

$$1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \cdots = \sum_{k=0}^{\infty} \frac{(-x)^k}{k+1}$$

1. Does the series converge when $x = 1$?

   When $x = 1$, we have

$$P(1) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1}.$$

   This is an alternating series, and the alternating series test immediately tells us the series $P(1)$ converges.

2. Does the series converge when $x = -1$?

When $x = -1$, we have

$$P(-1) = \sum_{k=0}^{\infty} \frac{(1)^k}{k+1}.$$

Positive-term series: use the Integral Test (if it applies)

$a(x) = 1/(x+1)$ is $+$, cont., $\downarrow$ on $[0, \infty)$, so Int Test applies

$\displaystyle\int_0^{\infty} \frac{1}{x+1}\,dx$ diverges, and so we know the series $P(-1)$ does also.

(By the way, this also tells us that the first series we looked at, $P(1)$ only converges conditionally, not absolutely.)

3. Does the series converge when $x = \frac{1}{2}$?

$$P(1/2) = \sum_{k=0}^{\infty} \frac{(-1/2)^k}{k+1} = \sum_{k=0}^{\infty} (-1)^k \frac{1}{2^k(k+1)}.$$

By the alternating series test, this series converges.

In fact, it converges absolutely, which we can conclude by comparing $\displaystyle\sum_{k=0}^{\infty} \frac{1}{2^k(k+1)}$ to the larger convergent series $\displaystyle\sum_{k=0}^{\infty} \frac{1}{2^k}$.

4. For what values of x does $P(x)$ converge absolutely? (Remember: Try the Ratio Test)

$$\begin{aligned} L &= \lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\frac{|x|^{k+1}}{k+2}\cdot\frac{k+1}{|x|^k} \\ &= \lim_{k\to\infty}|x|\cdot\frac{k+1}{k+2} = |x| \end{aligned}$$

Therefore the series converges absolutely whenever $|x| < 1$, that is, when $-1 < x < 1$, and diverges when $|x| > 1$.

Although the Ratio Test is inconclusive when $x = \pm 1$, we already figured out what the series does for those two possible values of x.

Thus $\displaystyle\sum_{k=0}^{\infty}\frac{(-x)^k}{k+1}$ converges on the interval $-1 < x \leq 1$, converging absolutely on the interior and conditionally at $x = 1$.

Find the interval of convergence for $\sum_{k=1}^{\infty} \frac{(x-1)^k}{k2^k}$

- **Use the Ratio Test to find where we have absolute convergence and most of the values of x where we have divergence:**

$$\begin{aligned} L &= \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \frac{|x-1|^{k+1}}{(k+1)2^{k+1}} \cdot \frac{k2^k}{|x-1|^k} \\ &= \lim_{k\to\infty} \frac{k|x-1|}{2(k+1)} = \frac{|x-1|}{2} \lim_{k\to\infty} \frac{k}{k+1} = \frac{|x-1|}{2} \end{aligned}$$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{(x-1)^k}{k2^k} \text{ conv's absolutely if } \frac{|x-1|}{2} < 1, \text{ div's if } \frac{|x-1|}{2} > 1.$$

$$\frac{|x-1|}{2} < 1 \Leftrightarrow |x-1| < 2 \Leftrightarrow -2 < x-1 < 2 \Leftrightarrow -1 < x < 3.$$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{(x-1)^k}{k2^k} \text{ conv's abs'ly if } -1 < x < 3; \text{ div's if } x < -1 \text{ or } x > 3.$$

Now know $\displaystyle\sum_{k=1}^{\infty} \frac{(x-1)^k}{k2^k}$ converges absolutely if $-1 < x < 3$; diverges if $x < -1$ or $x > 3$.

But what happens *at* $x = -1$, $x = 3$?

- **Check endpoints:**

  • When $x = 3$, the series we're dealing with is $\displaystyle\sum_{k=1}^{\infty} \frac{2^k}{k2^k}$.

  Simplifies to the harmonic series $\Rightarrow$ series diverges at x=3.

  • When $x = -1$, the series we're dealing with is $\displaystyle\sum_{k=1}^{\infty} \frac{(-2)^k}{k2^k}$.

  Simplifies to the $\sum \frac{(-1)^k}{k}$ $\Rightarrow$ series converges conditionally at $x = -1$.

  Therefore we know the power series converges on $[-1, 3)$, with absolute convergence on the interior of the interval.