Find the interval of convergence for the following power series:

1. $\displaystyle\sum_{j=0}^{\infty} \frac{x^j}{j!}$

- Using the ratio test, find the int. of conv., give or take endpoints:

$$\begin{aligned}\lim_{j\to\infty}\left|\frac{a_{j+1}}{a_j}\right| &= \lim_{j\to\infty}\frac{|x|^{j+1}j!}{|x|^j(j+1)!} \\ &= \lim_{j\to\infty}\frac{|x|}{j+1} \\ &= 0\end{aligned}$$

Since $\displaystyle\lim_{j\to\infty}\left|\frac{a_{j+1}}{a_j}\right| = 0 < 1$ **independent of x**, $\displaystyle\sum_{j=0}^{\infty}\frac{x^j}{j!}$ *always* converges absolutely, no matter what value of x you may choose to use.

Conclusion: The interval of convergence for $\displaystyle\sum_{k=0}^{\infty}\frac{x^k}{k!}$ is $(-\infty, \infty)$.

2. $\displaystyle\sum_{n=0}^{\infty}(n+1)(x-3)^n$

- Using the ratio test, find the int. of conv., give or take endpoints:

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\frac{(n+2)|x-3|^{n+1}}{(n+1)|x-3|^n} \\
&= \lim_{n\to\infty}\frac{(n+2)|x-3|}{n+1} \\
&= |x-3|\lim_{n\to\infty}\frac{n+2}{n+1} \\
&= |x-3|
\end{aligned}$$

$\Rightarrow \displaystyle\sum_{n=0}^{\infty}(n+1)(x-3)^n$ converges absolutely if $|x-3|<1$, diverges if $|x-3|>1$, i.e. converges absolutely when $-1<x-3<1$, or $2<x<4$, and diverges when $x<2$ or $x>4$.

2. (continued) $\sum_{n=0}^{\infty}(n+1)(x-3)^n$ converges absolutely when $2 < x < 4$, and diverges when $x < 2$ or $x > 4$.

But what happens at $x = 2$ and $x = 4$?

The ratio test is inconclusive, so check these cases individually.

- When $x = 4$, the series we're dealing with is $\sum_{n=0}^{\infty}(n+1)(1)^n = \sum_{n=0}^{\infty}(n+1)$.
  This series obviously diverges.
- When $x = 2$, the series we're dealing with is $\sum_{n=0}^{\infty}(-1)^n(n+1)$.
  Using the alternating series test, this series obviously diverges also.

Therefore, the interval of convergence is $(2, 4)$.

Find Taylor Series about $x_0 = 0$ for the following:

1. $f(x) = \sin(x)$

k	$f^{(k)}(x)$	$f^{(k)}(x_0)$	a_k
0	$\sin(x)$	0	$a_0 = 0/0! = 0$
1	$\cos(x)$	1	$a_1 = 1/1! = 1$
2	$-\sin(x)$	0	$a_2 = 0/2! = 0$
3	$-\cos(x)$	-1	$a_3 = -1/3!$
4	$\sin(x)$	0	$a_4 = 0$
5	$\cos(x)$	1	$a_5 = 1/5!$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

$\Rightarrow$ Taylor series for $\sin(x)$ based at $x_0 = 0$ is

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

1. (continued)

Write this Taylor series for $\sin(x)$ in sigma notation:

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

- Only odd powers of $x \Rightarrow$ write as x^{2k+1} or x^{2k-1}.
  I choose x^{2k+1}.
- Divide by that same number, factorial $\Rightarrow \dfrac{x^{2k+1}}{(2k+1)!}$
- What k do we need to start with? $\dfrac{x^{2k+1}}{(2k+1)!} = x$ when $k = 0$.
- Alternating sum $\Rightarrow (-1)^k$ or $(-1)^{k+1}$.
  Starting with $k = 0$,first term $= x$ (not $-x$) $\Rightarrow (-1)^k$.

Therefore, the power series for $\sin(x)$ is $\displaystyle\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!}$.

Find Taylor Series about $x_0 = 0$ for the following:

2. $f(x) = \cos(x)$

$\cos(x) = \dfrac{d}{dx}(\sin(x)) \Rightarrow$ differentiate the power series for $\sin(x)$.

Power series for $\sin(x)$ based at $x_0 = 0$:

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!},$$

$\Rightarrow$ Power series for $\cos(x)$ based at $x_0 = 0$:

$$\frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) \quad \text{or} \quad \frac{d}{dx}\left(\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}\right)$$

$$1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \cdots \quad \text{or} \quad \sum_{k=0}^{\infty} (-1)^k \frac{(2k+1)x^{2k}}{(2k+1)!}$$

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad \text{or} \quad \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$