Recall: If $f(x)$ has infinitely many derivatives at $x = x_0$, then the Taylor series for f centered (or based) at x_0 is

$$\sum_{k=0}^{\infty} a_k(x - x_0)^k \qquad \text{where} \qquad a_k = \frac{f^{(k)}(x_0)}{k!}$$

Find Taylor Series about $x_0 = 0$ for the following:

1. $f(x) = \sin(x)$
2. $g(x) = \cos(x)$
   Hint: $\frac{d}{dx}\sin(x) = \cos(x)$

We found that the Taylor series for $\sin(x)$ based at $x_0 = 0$ is

$$\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

But we're left with two crucial questions:

1. Where does this Taylor series converge?
2. Does this Taylor series actually converge to $\sin(x)$?

Previous Version of Taylor's Theorem:

Suppose that f is repeatedly differentiable on an interval I containing x_0 and that

$$P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n$$

is the nth order Taylor polynomial based at x_0. Suppose that for all x in I,

$$\left| f^{(n+1)}(x) \right| \leq K_{n+1}.$$

Then

$$\left| f(x) - P_n(x) \right| \leq \frac{K_{n+1}}{(n+1)!} \left| x - x_0 \right|^{n+1}.$$

Find power series expansions about $x_0 = 0$ for the following:

1. $f(x) = \sin(x)$
2. $g(x) = \cos(x)$
   Hint: $\frac{d}{dx}\sin(x) = \cos(x)$
3. $h(x) = \cos(x^2)$
   Feel free to use the result from (2).
4. $$H(x) = \int \cos(x^2)\, dx$$
   Then approximate $\displaystyle\int_0^1 \cos(x^2)\, dx$ accurate within 10^{-5}.