Find Taylor Series about $x_0 = 0$ for the following:

1. $f(x) = \sin(x)$

| $k$ | $f^{(k)}(x)$ | $f^(k)(x_0)$ | $a_k$ |
|---|---|---|---|
| 0 | $\sin(x)$ | 0 | $a_0 = 0/0! = 0$ |
| 1 | $\cos(x)$ | 1 | $a_1 = 1/1! = 1$ |
| 2 | $-\sin(x)$ | 0 | $a_2 = 0/2! = 0$ |
| 3 | $-\cos(x)$ | $-1$ | $a_3 = -1/3!$ |
| 4 | $\sin(x)$ | 0 | $a_4 = 0$ |
| 5 | $\cos(x)$ | 1 | $a_5 = 1/5!$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

$\Rightarrow$ Taylor series for $\sin(x)$ based at $x_0 = 0$ is

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

1. (continued)

Write this Taylor series for $\sin(x)$ in sigma notation:

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

- Only odd powers of $x \Rightarrow$ write as $x^{2k+1}$ or $x^{2k-1}$.
  I choose $x^{2k+1}$.
- Divide by that same number, factorial $\Rightarrow \dfrac{x^{2k+1}}{(2k+1)!}$
- What $k$ do we need to start with? $\dfrac{x^{2k+1}}{(2k+1)!} = x$ when $k = 0$.
- Alternating sum $\Rightarrow (-1)^k$ or $(-1)^{k+1}$.
  Starting with $k = 0$,first term $= x$ (not $-x$) $\Rightarrow (-1)^k$.

Therefore, the power series for $\sin(x)$ is $\displaystyle\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!}$.

Find Taylor Series about $x_0 = 0$ for the following:

2. $f(x) = \cos(x)$

$\cos(x) = \frac{d}{dx}(\sin(x)) \Rightarrow$ differentiate the power series for $\sin(x)$.

Power series for $\sin(x)$ based at $x_0 = 0$:

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!},$$

$\Rightarrow$ Power series for $\cos(x)$ based at $x_0 = 0$:

$$\frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) \quad \text{or} \quad \frac{d}{dx}\left(\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}\right)$$

$$1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \cdots \quad \text{or} \quad \sum_{k=0}^{\infty} (-1)^k \frac{(2k+1)x^{2k}}{(2k+1)!}$$

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad \text{or} \quad \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

We found that the power series for $\sin(x)$ based at $x_0 = 0$ is

$$\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

But we're left with two crucial questions:

1. Where does this Taylor series converge
2. Does this Taylor series actually converge to $\sin(x)$?

Find Taylor Series about $x_0 = 0$ for the following:

3. $\cos(x^2)$

   Feel free to use the result from (b).

   Since (on the interval of convergence), $\cos(x)$ actually **equals** $\displaystyle\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k)!}$, they are just two different ways of writing the same function.

   Thus

$$\cos(x^2) = \sum_{k=0}^{\infty}(-1)^k \frac{(x^2)^{2k}}{(2k)!} = \sum_{k=0}^{\infty}(-1)^k \frac{x^{4k}}{(2k)!} = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \ldots.$$

Find Taylor Series about $x_0 = 0$ for the following:

4. $\displaystyle\int \cos(x^2)\, dx$

   Then approximate $\displaystyle\int_0^1 \cos(x^2)\, dx$ accurate within $10^{-5}$.

$$\begin{aligned}
\cos(x^2) &= \sum_{k=0}^{\infty}(-1)^k \frac{x^{4k}}{(2k)!} = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \dots \\
\Rightarrow \int \cos(x^2)\, dx &= \int 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \dots\ dx \\
&= x - \frac{x^5}{5 \cdot 2!} + \frac{x^9}{9 \cdot 4!} - \frac{x^{13}}{13 \cdot 6!} + \dots + C \\
&= C + \sum_{k=0}^{\infty}(-1)^k \frac{x^{4k+1}}{(4k+1)\cdot(2k)!}
\end{aligned}$$

4. (continued)

Approximate $\int_0^1 \cos(x^2)\, dx$ accurate within $10^{-5}$.

$$\begin{aligned} \int \cos(x^2)\, dx &= C + \sum_{k=0}^{\infty} (-1)^k \frac{x^{4k+1}}{(4k+1)\cdot(2k)!} \\ \Rightarrow \int_0^1 \cos(x^2)\, dx &= \sum_{k=0}^{\infty} (-1)^k \frac{1}{(4k+1)\cdot(2k)!} - \sum_{k=0}^{\infty} 0 \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{1}{(4k+1)\cdot(2k)!} \end{aligned}$$

Alternating series, sequence of terms decreasing, so Alternating Series Test applies. $\left|\int_0^1 \cos(x^2)\, dx - S_N\right| \leq a_{N+1}$, so find $N$ so

$$\frac{1}{\big(4(N+1)+1\big)\cdot\big(2(N+1)\big)!} \leq 10^{-5}.$$

4. Approximate $\int_0^1 \cos(x^2)\,dx$ accurate within $10^{-5}$. (continued)

$\left|\int_0^1 \cos(x^2)\,dx - S_N\right| \leq a_{N+1}$, so find $N$ so

$$\begin{aligned}
\frac{1}{\big(4(N+1)+1\big)\cdot\big(2(N+1)\big)!} &\leq 10^{-5} \\
\frac{1}{(4N+5)\cdot(2N+2)!} &\leq 10^{-5}
\end{aligned}$$

Through experimentation, I find that

$$\begin{aligned}
\frac{1}{(4\cdot 2+5)\cdot(2\cdot 2+2)!} &= 0.00010684 \not\leq 10^{-5} \\
\frac{1}{(4\cdot 3+5)\cdot(2\cdot 3+2)!} &= 0.000001459 \leq 10^{-5}
\end{aligned}$$

So $\int_0^1 \cos(x^2)\,dx = \sum_{k=0}^{3} (-1)^k \frac{1}{(4k+1)\cdot(2k)!} \pm 10^{-5}$.