Reading Question:

Consider the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^4+5}$.

1. *Show that the series converges.*

   This series is not geometric, nor is it a p-series, so it is not one whose behavior we already know. The kth term test is inconclusive.

   Comparison Test: $\dfrac{1}{k^4+5} \leq \dfrac{1}{k^4}$ for all $k \geq 1$, so

   $$\sum_{k=1}^{\infty} \frac{1}{k^4+5} \leq \sum_{k=1}^{\infty} \frac{1}{k^4}, \; p\text{-series with } p = 4, \text{ so converges.}$$

   Thus by the comparison test, our smaller but still non-negative-term series converges.

Reading Question:

Consider the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^4+5}$.

2. *Estimate the limit of the series within* 0.01.

   Need $R_N \le 0.01$.

$$R_N \overset{def}{=} \sum_{k=N+1}^{\infty} \frac{1}{k^4+5} \underbrace{\le}_{comp} \sum_{k=N+1}^{\infty} \frac{1}{k^4}.$$

   If I find N so that $\displaystyle\sum_{k=N+1}^{\infty} \frac{1}{k^4} \le 0.01$, I will have found N so that $R_N \le 0.01$.

   There's no nice expression for this sum, ... but we *can* integrate the associated function.

Reading Question:

Consider the series $\sum_{k=1}^{\infty} \frac{1}{k^4+5}$.

2. *Estimate the limit of the series within* 0.01.(continued)

   By the Integral Test,

   $$\sum_{k=N+1}^{\infty} \frac{1}{k^4} \leq \int_N^{\infty} \frac{1}{x^4}\, dx,$$

   so

   $$R_N = \sum_{k=N+1}^{\infty} \frac{1}{k^4+5} \leq \sum_{k=N+1}^{\infty} \frac{1}{k^4} \leq \int_N^{\infty} \frac{1}{x^4}\, dx.$$

   If I can find N so that $\int_N^{\infty} \frac{1}{x^4}\, dx \leq 0.01$, then we'll have $R_N \leq 0.01$.

Reading Question:

Consider the series $\sum_{k=1}^{\infty} \frac{1}{k^4+5}$.

2. *Estimate the limit of the series within* 0.01 (continued)

$$\int_N^{\infty} \frac{1}{x^4}\,dx \leq 0.01 \quad \Rightarrow \quad -\frac{1}{3}\lim_{R\to\infty} \frac{1}{x^3}\bigg|_N^R \leq 0.01 \Rightarrow \frac{1}{3N^3} \leq 0.01$$

$$\Rightarrow \frac{100}{3} \leq N^3 \quad \Rightarrow \quad \left(\frac{100}{3}\right)^{\frac{1}{3}} \leq N \Rightarrow 3.2 \leq N$$

Thus $R_4 \leq 0.01$, so $S_4 = \sum_{k=1}^{4} \frac{1}{k^4+5}$ is within 0.01 of $\sum_{k=1}^{\infty} \frac{1}{k^4+5}$.
Using Maple, I find that $S_4 \approx 0.2297$
Thus $S = 0.2297 \pm 0.01$.

Reading Question:

Consider the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^4+5}$.

3. *Is your estimate an over- or under-estimate?*

   We found $S = S_4 \pm 0.01 = 0.2297 \pm 0.01$.

   S_4 itself will be an under-estimate because – since this is a positive-term series – every partial sum (including this one, S_4) will be less than the value of the series.

   So really, S lies between 0.2297 and 0.2397:

   $$0.2297 \leq S \leq 0.2397.$$

In Exercises 1-2, determine whether or not the series converges. If it converges, find upper and lower bounds on its limit.

1. $\displaystyle\sum_{m=1}^{\infty} \frac{1}{m\sqrt{1+m^2}}$

2. $\displaystyle\sum_{k=1}^{\infty} \frac{k}{(k^2+1)^2}$

In Exercises 3-5, determine whether or not the series converges or diverges. If the series converges, find a number N such that the partial sum S_N approximate the sum within 10^{-6} and then find S_N. If the series diverges, find a number N such that $S_N \geq 1000$.

3. $\displaystyle\sum_{j=1}^{\infty} \frac{1}{100+5j}$

4. $\displaystyle\sum_{k=0}^{\infty} \frac{k}{k^6+17}$

5. $\displaystyle\sum_{m=2}^{\infty} \frac{\ln(m)}{m^3}$