1. $\sum_{m=1}^{\infty} \frac{1}{m\sqrt{1+m^2}}$

This is on the supplement to PS 11, so I will not put the full solutions here.

I found that this is a convergent series; one set of upper and lower bounds is

$$\frac{1}{\sqrt{2}} \leq \sum_{m=1}^{\infty} \frac{1}{m\sqrt{1+m^2}} \leq 2,$$

although I could improve the lower bound quite a bit by using a better choice for a partial sum.

2. $\displaystyle\sum_{k=1}^{\infty} \frac{k}{(k^2+1)^2}$

Again, this is on the supplement to PS 11, so I will not include the full solution.

I found that this is a convergent series whose value lies in the interval

$$\frac{1}{4} \leq \sum_{k=1}^{\infty} \frac{k}{(k^2+1)^2} \leq \frac{1}{2}.$$

3. $\displaystyle\sum_{j=1}^{\infty} \frac{1}{100+5j}$

- **Is this a series I know?** Neither geometric nor p-series, so no.

  Notice: it's close to $\displaystyle\sum_{j=1}^{\infty} \frac{1}{5j}$, which diverges. Intuition tells me that given series will also diverge, but need to convince myself and others.

- ***j*th term test:** The jth term test is inconclusive.

- **Comparison Test vs Integral Test?**

  Beware the direction of comparison: $\displaystyle\sum_{j=1}^{\infty} \frac{1}{100+5j} \le \frac{1}{5}\sum_{j=1}^{\infty} \frac{1}{j} = \infty.$

  Not useful.

  Find a more useful comparison? Integral test?

  Can integrate the corresponding integral, *and* the integral test provides an easy way to deal with the approximation as well $\Rightarrow$ use the integral test.

3. $\displaystyle\sum_{j=1}^{\infty} \frac{1}{100+5j}$ (continued)

- **Integral test**: Determine the convergence/divergence of the associated integral $\displaystyle\int_1^{\infty} \frac{1}{100+5x}\,dx$.

  Let $u = 100 + 5x$, so $\frac{1}{5}\,du = dx$.

  $$\int_1^{\infty} \frac{1}{100+5x}\,dx = \frac{1}{5}\int_{105}^{\infty} \frac{1}{u}\,du, \text{ which diverges.}$$

  Thus $\displaystyle\sum_{j=1}^{\infty} \frac{1}{100+5j}$ diverges by the Integral Test.

3. $\displaystyle\sum_{j=1}^{\infty} \frac{1}{100+5j}$ (continued)

- **Since the series diverges, find N so $\mathbf{S_N \geq 1000}$:**

  That is, find N so that $\displaystyle\sum_{j=1}^{N} \frac{1}{100+5j} \geq 1000.$

  I don't know any general expression for this partial sum.
  Can I switch over to an integral?
  In the supplement to PS 10, we are showing that
  $\displaystyle\int_1^{n+1} a(x)\,dx \leq \sum_{k=1}^{\infty} a_k$ for continuous, non-negative, decreasing $a(x)$.
  Since $a(x) = \dfrac{1}{100+5x}$ is continuous non-negative and decreasing, I therefore know

$$\sum_{j=1}^{N} \frac{1}{100+5j} \geq \int_1^{N+1} \frac{1}{100+5x}\,dx.$$

  Thus, if we find N so the integral is larger than 1000, the partial sum will also be.

3. $\displaystyle\sum_{j=1}^{\infty} \frac{1}{100+5j}$ (continued)

- **Since the series diverges, find N so $\mathbf{S_N \geq 1000}$:**

  Using $u = 100 + 5x$, $\frac{1}{5}\, du = dx$, $x = 1 \Rightarrow u = 105$,
  $x = N + 1 \Rightarrow u = 100 + 5(N+1)$,

$$
\begin{aligned}
\int_1^{N+1} \frac{1}{100+5x}\, dx &\geq 1000 \\
\frac{1}{5}\int_{105}^{100+5(N+1)} \frac{1}{u}\, du &\geq 1000 \\
\ln(100+5(N+1)) - \ln(105) &\geq 5000 \\
\ln(100+5(N+1)) &\geq 5000 + \ln(105) \\
100+5(N+1) &\geq e^{5000+\ln(105)} = e^{5000}e^{\ln(105)} \\
100+5(N+1) &\geq 105e^{5000} \\
N+1 &\geq \frac{105e^{5000}-100}{5} = 21e^{5000} - 20 \\
N &\geq 21e^{5000} - 21 \approx 6.23 \times 10^{2172}
\end{aligned}
$$

4. $\displaystyle\sum_{k=0}^{\infty} \frac{k}{k^6+17}$

- **Is this a series I know?** Neither geometric nor p-series, so no.

  Notice: close to $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^5}$, which converges. Intuition says this series will converge, but must convince ourselves.

- ***k*th term test:** Inconclusive.

- **Comparison Test vs Integral Test?**

  Don't particularly care to integrate $\displaystyle\int_0^{\infty} \frac{x}{x^6+17}\,dx$, so try the **comparison test**

4. $\displaystyle\sum_{k=0}^{\infty} \frac{k}{k^6+17}$

- **Comparison Test:**

$$k^6+17 \geq k^6 \Rightarrow \frac{k}{k^6+17} \leq \frac{k}{k^6} = \frac{1}{k^5} \text{ for all } k \geq 1$$

Be careful! $\dfrac{k}{k^6+17}$ is defined for $k=0$, but $\dfrac{1}{k^5}$ is not.

$$\sum_{k=0}^{\infty} \frac{k}{k^6+17} = \frac{0}{0^6+17} + \sum_{k=1}^{\infty} \frac{k}{k^6+17} \quad \leq \quad \frac{0}{0^6+17} + \sum_{k=1}^{\infty} \frac{1}{k^5}$$

$$\Rightarrow \sum_{k=0}^{\infty} \frac{k}{k^6+17} \quad \leq \quad 0 + \sum_{k=1}^{\infty} \frac{1}{k^5}.$$

(Thus you have a_0 trailing along with you, but in this case it's 0.)

Because the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^5}$ is a p-series with $p = 5 > 1$, this series converges, and so our original series converges as well, by the comparison test.

4. $\displaystyle\sum_{k=0}^{\infty} \frac{k}{k^6+17}$

- **Finding N so S_N approximates S to within 10^{-6}:**

  I need to find N so that $R_N = \displaystyle\sum_{k=N+1}^{\infty} \frac{k}{k^6+17} \leq 10^{-6}$.

  If I can find N so that the larger remainder $\displaystyle\sum_{k=N+1}^{\infty} \frac{1}{k^5} \leq 10^{-6}$, then I'll be done.

  Unfortunately, our comparison series is not geometric.

  Bring the integral test into it, giving a string of inequalities:

  $$\underbrace{\sum_{k=N+1}^{\infty} \frac{k}{k^6+17}}_{R_N} \leq \sum_{k=N+1}^{\infty} \frac{1}{k^5} \leq \int_N^{\infty} \frac{1}{x^5}\,dx.$$

  If I can find N so that the integral is less than 10^{-6}, then of course my original R_N will be as well.

4. $\displaystyle\sum_{k=0}^{\infty} \frac{k}{k^6+17}$

- **Finding N so S_N approximates S to within 10^{-6}:** (continued)

$$\int_N^{\infty} \frac{1}{x^5}\,dx \leq 10^{-6} \quad \Rightarrow \quad \lim_{R\to\infty} -\frac{1}{4}x^{-4}\bigg|_N^R \leq 10^{-6} \Rightarrow 0 + \frac{1}{4N^4} \leq 10^{-6}$$

$$4N^4 \geq 10^6 \quad \Rightarrow \quad N^4 \geq \frac{10^6}{4} \Rightarrow N \geq (250000)^{1/4} \Rightarrow N \geq 22.4$$

Thus S_{23} is within 10^{-6} of $\displaystyle\sum_{k=0}^{\infty} \frac{k}{k^6+17}$.

Using Maple, I therefore can say that

$$\sum_{k=0}^{\infty} \frac{k}{k^6+17} = 0.08582562924 \pm 10^{-6}.$$

5. $\displaystyle\sum_{m=2}^{\infty} \frac{\ln(m)}{m^3}$

This problem is on the PS 11 supplement

The series converges, and we can show that $S = S_{1,000,000} \pm 10^{-6}$.