Determine whether or not the following alternating series converge. For those that converge, first find upper and lower bounds, and then approximate accurate to within 0.001.

1. $\displaystyle\sum_{k=2}^{\infty} \frac{(-1)^k}{\ln(k)}$

   - **Convergence?**

     Alternating Series Test, Part 1: Suppose $\lim_{k\to\infty} a_k = 0$ and $0 \le a_{k+1} \le a_k$ for all $k \ge 1$. Then the alternating series $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ converges.

     $\{a_k\} = \left\{\dfrac{1}{\ln(k)}\right\}$. $\ln(x)$ is positive, increasing on $[2, \infty) \Rightarrow \dfrac{1}{\ln(x)}$ positive,decreasing on $[2, \infty) \Rightarrow$ Alternating Series Test applies.

1. $\displaystyle\sum_{k=2}^{\infty} \frac{(-1)^k}{\ln(k)}$

- **Convergence?** (continued)

$\bullet$ if $\displaystyle\lim_{k\to\infty} \frac{1}{\ln(k)} = 0$, $\displaystyle\sum_{k=2}^{\infty} \frac{(-1)^k}{\ln(k)}$ converges (by A.S.T.)

$\bullet$ if $\displaystyle\lim_{k\to\infty} \frac{1}{\ln(k)} \neq 0$, $\displaystyle\sum_{k=2}^{\infty} \frac{(-1)^k}{\ln(k)}$ diverges, by kth term test.

Since $\displaystyle\lim_{k\to\infty} \frac{1}{\ln(k)} = 0$, $\displaystyle\sum_{k=2}^{\infty} \frac{(-1)^k}{\ln(k)}$ does indeed converge.

1. $\displaystyle\sum_{k=2}^{\infty} \frac{(-1)^k}{\ln(k)}$

- **Upper and Lower Bounds:**

  Alternating Series Test, Part 2: If an alternating series converges, its limit lies between any two consecutive partial sums. That is, if the series converges to S, then S lies between S_N and S_{N+1} for any N.

  Pick any 2 consecutive partial sums.

  Easiest: S_2 and S_3. Because S_2 is positive and $S_3 = S_2 -$something,

$$\begin{aligned} S_3 &\leq S \leq S_2 \\ \frac{1}{\ln(2)} - \frac{1}{\ln(3)} &\leq S \leq \frac{1}{\ln(2)} \\ .532 &\leq S \leq 1.443 \end{aligned}$$

1. $\displaystyle\sum_{k=2}^{\infty} \frac{(-1)^k}{\ln(k)}$

- **Approximate within** 0.001

  Alternating Series Test, Part 3: If an alternating series converges, $|S - S_N| \leq a_{N+1}$.

  If $a_{N+1} \leq 0.001$, this will guarantee $|S - S_N| \leq 0.001$.

  Remember, $a_k = \dfrac{1}{\ln(k)}$.

  $$\frac{1}{\ln(N+1)} \leq \frac{1}{1000} \Rightarrow \ln(N+1) \geq 1000 \Rightarrow N+1 \geq e^{1000} \Rightarrow N \geq e^{1000} - 1$$

  Let M=the next integer larger than $e^{1000} - 1$.
  $\displaystyle\sum_{k=2}^{M} \frac{(-1)^k}{\ln(k)}$ is within 0.001 of $\displaystyle\sum_{k=2}^{\infty} \frac{(-1)^k}{\ln(k)}$.

2. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{n^2-1}$

- **Convergence?**

  $\{a_n\} = \left\{\dfrac{n^2}{n^2-1}\right\}.$

  Looking at a graph of $\dfrac{x^2}{x^2-1}$, I can see that it is positive and decreasing, so the Alternating Series Test applies.

  - If $\displaystyle\lim_{n\to\infty}\frac{n^2}{n^2-1}=0$, $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n^2}{n^2-1}$ converges (by the A.S.T.)
  - If $\displaystyle\lim_{n\to\infty}\frac{n^2}{n^2-1}\neq 0$, $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n^2}{n^2-1}$ diverges, by the nth term test.

  Since $\displaystyle\lim_{n\to\infty}\frac{n^2}{n^2-1}=1\neq 0$, $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n^2}{n^2-1}$ diverges

3. $\displaystyle\sum_{j=3}^{\infty} \frac{(-1)^j}{4^j}$

- **Convergence?**

$\{a_j\} = \left\{\frac{1}{4^j}\right\}$. Because 4^x is positive and increasing on $[3, \infty)$, $\frac{1}{4^x}$ is positive and decreasing on $[3, \infty)$, so the Alternating Series Test applies.

- If $\displaystyle\lim_{j\to\infty} \frac{1}{4^j} = 0$, $\displaystyle\sum_{j=3}^{\infty} \frac{(-1)^j}{4^j}$ converges (by the A.S.T.)
- If $\displaystyle\lim_{j\to\infty} \frac{1}{4^j} \neq 0$, $\displaystyle\sum_{j=3}^{\infty} \frac{(-1)^j}{4^j}$ diverges, by the jth term test.

Since $\displaystyle\lim_{j\to\infty} \frac{1}{4^j} = 0$, $\displaystyle\sum_{j=3}^{\infty} \frac{(-1)^j}{4^j}$ does indeed converge.

3. $\displaystyle\sum_{j=3}^{\infty} \frac{(-1)^j}{4^j}$

- **Upper and Lower Bounds:**

Pick any 2 consecutive partial sums.

Because S_3 is negative and $S_4 = S_3 +$something,

$$\begin{aligned} S_3 &\leq S \leq S_4 \\ -\frac{1}{4^3} &\leq S \leq -\frac{1}{4^3} + \frac{1}{4^4} \\ -.0156 &\leq S \leq -.0118 \end{aligned}$$

3. $\displaystyle\sum_{j=3}^{\infty} \frac{(-1)^j}{4^j}$

- **Approximate within** 0.001

  If $a_{N+1} \leq 0.001$, this will guarantee $|S - S_N| \leq 0.001$.

  Remember, $a_j = \dfrac{1}{4^j}$.

$$\frac{1}{4^{N+1}} \leq \frac{1}{1000} \quad \Rightarrow \quad 4^{N+1} \geq 1000 \Rightarrow (N+1)\ln(4) \geq \ln(1000)$$

$$\Rightarrow \quad N \geq \frac{\ln(1000)}{\ln(4)} - 1 \approx 3.98.$$

  Thus $\displaystyle\sum_{k=3}^{4} \frac{(-1)^j}{4^j}$ is within 0.001 of $\displaystyle\sum_{j=3}^{\infty} \frac{(-1)^j}{4^j}$, so

$$S \approx S_4 \pm 0.001 \approx 0.011718 \pm 0.001.$$

1. $\displaystyle\sum_{j=0}^{\infty}(-1)^j \frac{e^j}{3^{j+1}+j}$

- **Convergence? Use Alternating Series Test/jth Term Test:**

$$0 \leq \lim_{j\to\infty} \frac{e^j}{3^{j+1}+j} \leq \lim_{j\to\infty} \frac{1}{3}\left(\frac{e}{3}\right)^j = 0, \text{ since } \frac{e}{3} < 1.$$

Squeeze Principle $\Rightarrow \displaystyle\lim_{j\to\infty} \frac{e^j}{3^{j+1}+j} = 0 \Rightarrow$ the alternating series $\displaystyle\sum_{j=0}^{\infty}(-1)^j \frac{e^j}{3^{j+1}}$ converges by the Alternating Series Test.

But does it converge *conditionally* or *absolutely*?

1. $\sum_{j=0}^{\infty}(-1)^j \frac{e^j}{3^{j+1}+j}$ (continued)

- **Conditional vs Absolute Convergence?**

  Does $\sum_{j=0}^{\infty}\left|(-1)^j \frac{e^j}{3^{j+1}+j}\right| = \sum_{j=0}^{\infty} \frac{e^j}{3^{j+1}+j}$ converge?

  Comparison Test vs Integral Test: I don't particularly feel like integrating $\frac{3^x}{3^{x+1}+1}$, so try comparison test.

  Since $\sum_{j=0}^{\infty} \frac{e^j}{3^{j+1}+j} \leq \sum_{j=0}^{\infty} \frac{e^j}{3^{j+1}} = \frac{1}{3}\sum_{j=0}^{\infty}\left(\frac{e}{3}\right)^j$, which is a convergent geometric series, $\sum_{j=0}^{\infty} \frac{e^j}{3^{j+1}+j}$ converges.

  Hence $\sum_{j=0}^{\infty}(-1)^j \frac{e^j}{3^{j+1}+j}$ **converges absolutely.**

2. $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{2k+2}{k^2+2k}$

- **Convergence? Use Alternating Series Test/kth Term Test:**

$$\lim_{k\to\infty}\frac{2k+2}{k^2+2k}\overset{\text{l'Hôp}}{=}0$$

Therefore the alternating series $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{2k+2}{k^2+2k}$ converges by the alternating series test.

2. $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{2k+2}{k^2+2k}$ (continued)

- **Conditional vs Absolute Convergence?**

  Does $\sum_{k=1}^{\infty}\left|(-1)^{k+1}\frac{2k+2}{k^2+2k}\right| = \sum_{k=1}^{\infty}\frac{2k+2}{k^2+2k}$ converge?

  Comparison test vs Integral Test: The most obvious things to compare the numerator and denominator to get me nowhere, since $2k+2 \geq 2k$, but $\frac{1}{k^2+2k} \leq \frac{1}{k^2}$.
  On the other hand, this can be easily integrated:

  $$\int_1^{\infty}\frac{2x+2}{x^2+2x}\,dx = \int_{x=1}^{\infty}\frac{1}{u}\,du = \ln(x^2+2x)\Big|_1^{\infty}.$$

  This diverges, so $\sum_{k=1}^{\infty}\left|(-1)^{k+1}\frac{2k+2}{k^2+2k}\right|$ diverges, and our original sum **converges conditionally.**