1. Determine whether the following series converge conditionally, converge absolutely, or diverge. For those that converge, find upper and lower bounds.

(a) $\displaystyle\sum_{j=1}^{\infty} \frac{(-1)^j}{\sqrt{2j}(j+1)}$

- **Convergence? Alternating series test/jth term test:**
  Graph of $\dfrac{\sqrt{2x}}{x+1}$ positive, $\downarrow$, so the alternating series test applies:
  $\displaystyle\lim_{j\to\infty} \frac{1}{\sqrt{2j}(j+1)} = 0 \Rightarrow$ by the A.S.T., $\displaystyle\sum_{j=1}^{\infty} \frac{(-1)^j}{\sqrt{2j}(j+1)}$ does converge.
- **Absolute? Conditional?** Use **Comparison Test:**

$$\sum_{j=1}^{\infty} \left| \frac{(-1)^j}{\sqrt{2j}(j+1)} \right| \le \sum_{j=1}^{\infty} \frac{1}{j^{3/2}}, \text{ which converges.}$$

$$\Rightarrow \sum_{j=1}^{\infty} \left| \frac{(-1)^j}{\sqrt{2j}(j+1)} \right| \text{ converges} \Rightarrow \sum_{j=1}^{\infty} \frac{(-1)^j}{\sqrt{2j}(j+1)} \text{ converges absolutely.}$$

1(a) $\sum_{j=1}^{\infty} \frac{(-1)^j}{\sqrt{2j}(j+1)}$ (continued)

- **Upper and lower bounds:**
  Because this is an alternating series whose positive-terms are positive and decreasing, can use any two consecutive partial sums:

$$S_3 \approx -.2889487964 \quad \leq S \leq \quad S_2 \approx -.1868867238$$

or

$$S_{101} \approx -.2483703820 \quad \leq S \leq \quad S_{100} \approx -.2476805805$$

1(b) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^5}{n^6+17}$

- **Convergence? Alternating Series Test/nth term test:**

  Graph of $\frac{x^5}{x^6+17}$ always positive on $[1,\infty)$, but *increases* on $[1, 2.1]$ or so, and then decreases from then on out.

  To apply the Alternating Series Test, we thus need to break up the sum into

  $$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^5}{n^6+17}=\sum_{n=1}^{3}(-1)^{n+1}\frac{n^5}{n^6+17}+\sum_{n=4}^{\infty}(-1)^{n+1}\frac{n^5}{n^6+17}.$$

  $\lim_{n\to\infty}\frac{n^5}{n^6+17}=0$, so by the A.S.T., $\sum_{n=4}^{\infty}(-1)^{n+1}\frac{n^5}{n^6+17}$ converges,

  and so (since $\sum_{n=1}^{3}(-1)^{n+1}\frac{n^5}{n^6+17}$ is clearly finite), $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^5}{n^6+17}$ converges.

1(b) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^5}{n^6+17}$ (continued)

- **Absolute vs Conditional?**

  Use the Integral Test on $\displaystyle\sum_{n=1}^{\infty}\left|(-1)^{n+1}\frac{n^5}{n^6+17}\right|$, using $u = x^6 + 17$, so $\frac{1}{6}\,du = x^5\,dx$, $x = 1 \Rightarrow u = 18$, $x = \infty \Rightarrow u = \infty$:

  $$\int_1^{\infty}\frac{x^5}{x^6+17}\,dx = \frac{1}{6}\int_{18}^{\infty}\frac{1}{u}\,du, \text{ which diverges.}$$

  Since the integral diverges, $\displaystyle\sum_{n=1}^{\infty}\left|(-1)^{n+1}\frac{n^5}{n^6+17}\right|$ diverges as well by the Integral test.

  Thus $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^5}{n^6+17}$ **converges conditionally.**

1(b) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^5}{n^6+17}$ (continued)

- **Upper and Lower Bounds:**
  Because this is an alternating series, I can use any two consecutive partial sums as upper and lower bounds:

$$S_2 \leq \sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^5}{n^6+17} \leq S_1 \overset{\text{Maple}}{\Rightarrow} -\frac{55}{162} \leq \sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^5}{n^6+17} \leq \frac{1}{18}.$$

1(c) $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{k^2}{k^2-5k}$

- **Convergence? Alternating Series Test/kth term test:**

  Looking at the graph, we can see that $\frac{x^2}{x^2-5x}$ is a positive, decreasing function, so the A.S.T. applies. However, we don't even need it:

$$\lim_{k\to\infty}\frac{k^2}{k^2-5k}\overset{\text{l'Hôp}}{=}\frac{2k}{2k}=1\neq 0$$

  Since the sequence of *terms* converges to something *other than 0*, the series diverges, by the kth term test.

(d) $\sum_{k=1}^{\infty} \frac{\cos(k)}{k^4+1}$

This is not an alternating series! However, it also is not a positive-term series.

- **Convergence?**
  - **kth term test:**

$$\lim_{k\to\infty} \frac{\cos(k)}{k^4+1} = 0 \Rightarrow \text{ Inconclusive!}$$

  - **Alternating Series Test?** Does not apply - series isn't alternating
  - **Integral Test?** Does not apply - terms aren't all non-negative
  - **Comparison Test?** Technically, does not apply - terms aren't all non-negative. However, we can compare to something both above *and* below. Rather than doing that however, let's use a cool feature of absolute convergence:

    I can tell practically by looking at it that this is going to converge absolutely... *So ... we'll wait to answer this question!*

1(d) $\displaystyle\sum_{k=1}^{\infty} \frac{\cos(k)}{k^4+1}$ (continued)

- **Absolute Convergence?**
  Does $\displaystyle\sum_{k=1}^{\infty} \left|\frac{\cos(k)}{k^4+1}\right|$ converge?

  $$\sum_{k=1}^{\infty} \left|\frac{\cos(k)}{k^4+1}\right| \le \sum_{k=1}^{\infty} \frac{1}{k^4+1} \le \sum_{k=1}^{\infty} \frac{1}{k^4}, \text{ which converges.}$$

  Thus $\displaystyle\sum_{k=1}^{\infty} \left|\frac{\cos(k)}{k^4+1}\right|$ converges by the Comparison Test, and so the original series $\displaystyle\sum_{k=1}^{\infty} \frac{\cos(k)}{k^4+1}$ **converges absolutely**.
- **So... Convergence?**
  Since a sum of absolute values is always greater than the absolute value of the sum ... **yes!**

1(d) $\sum_{k=1}^{\infty} \frac{\cos(k)}{k^4+1}$ (continued)

- **Upper and Lower Bounds:**
Since this isn't an alternating series, can't use any two consecutive partial sums as bounds.
Go back to the comparison we used to show absolute convergence.

$$-\sum_{k=1}^{\infty} \frac{1}{k^4} \leq \sum_{k=1}^{\infty} \frac{\cos(k)}{k^4+1} \leq \sum_{k=1}^{\infty} \frac{1}{k^4}.$$

We can find an upper bound for the sum on the right using the integral test.

$$\sum_{k=1}^{\infty} \frac{1}{k^4} \leq 1 + \int_1^{\infty} \frac{1}{x^4}\, dx = 1 + \frac{1}{3} = \frac{4}{3}.$$

Therefore

$$-\frac{4}{3} \leq \sum_{k=1}^{\infty} \frac{\cos(k)}{k^4+1} \leq \frac{4}{3}.$$

2.Determine whether the following series converge conditionally, converge absolutely, or diverge. For those that converge, approximate each to within 10^{-6}.

(a) $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k^2+1}$

- **Convergence? Alternating Series Test/kth term test:**

  Graph of $\dfrac{1}{x^2+1}$ positive, $\downarrow$, so A.S.T. applies.

  It's easy enough to see that this series not only converges by the A.S.T. (the limit of the kth term is 0), but converges absolutely $\left(\sum\left|(-1)^{k+1}\dfrac{1}{k^2+1}\right| \leq \sum \dfrac{1}{k^2}\right)$.

2(a) $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k^2+1}$ (continued)

- In order for $|S - S_N| \le 10^{-6}$, need $a_{N+1} \le 10^{-6}$.

$$\begin{aligned}
\frac{1}{(N+1)^2+1} &\le 10^{-6} \\
(N+1)^2+1 &\ge 10^{6} \\
N+1 &\ge \sqrt{999,999} \\
N &\ge 999.9995
\end{aligned}$$

Thus S_{1000} is within 10^{-6} of S. Using Maple,

$$S = .3639850 \pm 10^{-6}.$$

2(b) $\sum_{k=2}^{\infty} \frac{7 - \sin(k)}{k^2 + 14k}$

Not an alternating series! In fact, it's a positive-valued series!

- **Convergence?**

$$\sum_{k=2}^{\infty} \frac{7 - \sin(k)}{k^2 + 14k} \leq \sum_{k=2}^{\infty} \frac{8}{k^2 + 14k} \leq \sum_{k=2}^{\infty} \frac{8}{k^2}.$$

$\sum_{k=2}^{\infty} \frac{8}{k^2}$ converges $\Rightarrow \sum_{k=2}^{\infty} \frac{7 - \sin(k)}{k^2 + 14k}$ converges, by the Comparison Test.

2(b) $\sum_{k=2}^{\infty} \frac{7-\sin(k)}{k^2+14k}$ (continued)

- **Find N so $|\mathbf{S}-\mathbf{S_N}| \leq \mathbf{10^{-6}}$.**

  If I find N so $\sum_{k=N+1}^{\infty} \frac{8}{k^2} \leq 10^{-6}$, then $\sum_{k=N+1}^{\infty} \frac{7-\sin(k)}{k^2+14k} \leq 10^{-6}$,

  and so $\sum_{2}^{N} \frac{7-\sin(k)}{k^2+14k}$ will be within 10^{-6} of $\sum_{k=2}^{\infty} \frac{7-\sin(k)}{k^2+14k}$.

  How can I make $\sum_{k=N+1}^{\infty} \frac{8}{k^2} \leq 10^{-6}$? Use the integral test.

  $$\sum_{k=N+1}^{\infty} \frac{8}{k^2} \leq \int_N^{\infty} \frac{8}{x^2}\,dx = \lim_{t\to\infty}\left(-\frac{8}{t}+\frac{8}{N+1}\right) = \frac{8}{N+1}$$

  So we want

  $$\frac{8}{N+1} \leq 10^{-6} \Rightarrow 8000000 \leq N+1 \Rightarrow N \geq 7999999.$$

  Thus S is within 10^{-6} of $S_{8000000}$.