2(c) $\displaystyle\sum_{k=5}^{\infty} \frac{k^4 + 400k^3}{1000k^4 + k}$

- This is not a series we recognize.
- When we try the kth term test,

$$\lim_{k\to\infty} \frac{k^4 + 400k^3}{1000k^4 + k} = \frac{1}{1000} \neq 0.$$

Therefore, while the sequence of terms converges to $1/1000$, the series $\displaystyle\sum_{k=5}^{\infty} \frac{k^4 + 400k^3}{1000k^4 + k}$ diverges by the kth term test.

3. Determine the absolute convergence, conditional convergence, or divergence of:

(a) $\displaystyle\sum_{j=1}^{\infty} \frac{(-1)^j}{j+e^j}$

- This is not a series we just recognize.
- **Alternating Series Test/*j*th Term Test**:
  - $x + e^x$ an increasing function $\Rightarrow \left\{ \frac{1}{j+e^j} \right\}$ is a decreasing sequence
  - $\displaystyle\lim_{j\to\infty} \frac{1}{j+e^j} = 0$

  Therefore $\displaystyle\sum_{j=1}^{\infty} \frac{(-1)^j}{j+e^j}$ converges, by the A.S.T.

3(a) $\displaystyle\sum_{j=1}^{\infty} \frac{(-1)^j}{j+e^j}$ (continued)

- **Conditional vs Absolute?** Does $\displaystyle\sum_{j=1}^{\infty} \frac{1}{j+e^j}$ converge?

  Comparison Test: $\displaystyle\sum_{j=1}^{\infty} \frac{1}{j+e^j} \leq \sum_{j=1}^{\infty} \frac{1}{j}$ and $\displaystyle\sum_{j=1}^{\infty} \frac{1}{e^j}$.

  Harmonic Series diverges $\Rightarrow \displaystyle\sum_{j=1}^{\infty} \frac{1}{j}$ is not a useful comparison.

  $\displaystyle\sum_{j=1}^{\infty} \frac{1}{e^j} = \sum_{j=1}^{\infty} \left(\frac{1}{e}\right)^j$ geom. series with $|r| < 1 \Rightarrow$ *useful* comparison.

  Larger series converges $\Rightarrow$ smaller series $\displaystyle\sum_{j=1}^{\infty} \frac{1}{j+e^j}$ also converge.

Thus $\displaystyle\sum_{j=1}^{\infty} \frac{(-1)^j}{j+e^j}$ **converges absolutely**

3(b) $\displaystyle\sum_{n=1}^{\infty} n\left(-\frac{2}{3}\right)^n$

- This is not a series I just recognize (although it's oh-so-close).
- **Alternating Series Test/*n*th Term Test:**

  • Looking at the graph of $x\left(-\frac{2}{3}\right)^x$, I see it is a decreasing function from about $x = 3$ on. Break into $\displaystyle\sum_{n=1}^{3} n\left(-\frac{2}{3}\right)^n + \sum_{n=4}^{\infty} n\left(-\frac{2}{3}\right)^n$; apply A.S.T. to $\displaystyle\sum_{n=4}^{\infty} n\left(-\frac{2}{3}\right)^n$.

  • $\displaystyle\lim_{n\to\infty} n\left(\frac{2}{3}\right)^n = \infty \cdot 0 \Rightarrow$ Indeterminate Form.

$$\lim_{n\to\infty} n\left(\frac{2}{3}\right)^n = \lim_{n\to\infty} \frac{n}{\left(\frac{3}{2}\right)^n} \overset{\text{l'Hôp}}{=} \lim_{n\to\infty} \frac{1}{\ln\left(\frac{3}{2}\right)\left(\frac{3}{2}\right)^n} = 0.$$

Thus the alternating series $\displaystyle\sum_{n=1}^{\infty} n\left(-\frac{2}{3}\right)^n$ converges.

3(b) $\sum_{n=1}^{\infty} n\left(-\frac{2}{3}\right)^n$ (continued)

- **Absolute vs Conditional?** Does $\sum_{n=1}^{\infty} n\left(\frac{2}{3}\right)^n$ converge?
  - **Comparison Test:**

    Most obvious comparison: $n\left(\frac{2}{3}\right)^n \geq \left(\frac{2}{3}\right)^n$, *not useful*

  - **Integral Test:** I could think about integrating $a(x) = x\left(\frac{2}{3}\right)^x$ using integration by parts, and will give it a shot if no test ends up flowing easily. But I would need to first break up sum as for the A.S.T., since need $a(x) = x\left(\frac{2}{3}\right)^x$ to be a continuous, positive, decreasing function.

  - **Ratio Test:** Since the terms are never 0, it applies.

    $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)\left(\frac{2}{3}\right)^{n+1}}{n\left(\frac{2}{3}\right)^n} = \lim_{n\to\infty}\frac{n+1}{n}\cdot\frac{\frac{2}{3}}{1} = \frac{2}{3} < 1.$$

    Since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$, the series **converges absolutely**.

3(c) $\displaystyle\sum_{m=2}^{\infty} \frac{(-1)^m m}{(m^2-1)^5}$

- **Alternating Series Test/mth Term Test:**

  • Looking at the graph of $\dfrac{x}{(x^2-1)^5}$, I see it is continuous, positive and decreasing on $[2, \infty)$.

  • $\displaystyle\lim_{m\to\infty} \frac{m}{m^2-1)^5} = 0$

  Thus by the A.S.T., the alternating series $\displaystyle\sum_{m=1}^{\infty} \frac{(-1)^m m}{(m^2-1)^5}$ converges.

- **Conditional vs Absolute?** Does $\displaystyle\sum_{m=2}^{\infty} \frac{m}{(m^2-1)^5}$ converge?

  - **Comparison Test?**
    Since $m^2 - 1 < m^2$, we have $\dfrac{m}{(m^2-1)^5} > \dfrac{m}{m^{10}}$, which is not a useful direction. There are certainly other comparisons we could try, but that was the easiest one.

3(c) $\sum_{m=2}^{\infty} \frac{(-1)^m m}{(m^2-1)^5}$ (continued)

- **Conditional vs Absolute?** Does $\sum_{m=2}^{\infty} \frac{m}{(m^2-1)^5}$ converge? (cont'd)
  - **Integral Test:** The integral test seems best here.

    Does it apply?

    $\frac{x}{(x^2-1)^5}$ is certainly positive and continuous (as the denominator is never 0).

    Is it decreasing on $[1, \infty)$?

    Could look at a graph, but that isn't always convincing. Here, I illustrate taking the derivative to see if it's always negative.

$$\begin{aligned} \frac{d}{dx}\left(\frac{x}{(x^2-1)^5}\right) &= \frac{(x^2-1)^5 \cdot 1 - x \cdot 5(x^2-1)^4(2x)}{((x^2-1)^5)^2} \\ &= \ldots = \frac{1-9x^2}{(x^2-1)^6} \leq 0 \text{ on } [1, \infty) \end{aligned}$$

Derivative negative $\Rightarrow a(x)$ decreasing $\Rightarrow$ integral test applies.

3(c) $\displaystyle\sum_{m=2}^{\infty} \frac{(-1)^m m}{(m^2-1)^5}$ (continued)

- **Conditional vs Absolute?** Does $\displaystyle\sum_{m=2}^{\infty} \frac{m}{(m^2-1)^5}$ converge? (cont'd)
  - **Integral Test:** (continued)

    Thus $\displaystyle\sum_{m=2}^{\infty} \left| \frac{(-1)^m m}{(m^2-1)^5} \right|$ does whatever $\displaystyle\int_2^{\infty} \frac{x}{(x^2-1)^5}\,dx$ does.

$$\begin{aligned}
\int_2^{\infty} \frac{x}{(x^2-1)^5}\,dx &= \lim_{R\to\infty} \int_2^{R} \frac{x}{(x^2-1)^5}\,dx = \lim_{R\to\infty} \frac{1}{2}\,\frac{(x^2-1)^{-4}}{-4}\bigg|_2^R \\
&= \lim_{R\to\infty} -\frac{1}{8(x^2-1)^4}\bigg|_2^R \\
&= \lim_{R\to\infty} -\frac{1}{8(R^2-1)^4} + \frac{1}{8(2^2-1)^4} = \frac{1}{8\cdot 81}
\end{aligned}$$

So the integral, and hence the series $\displaystyle\sum_{m=2}^{\infty} \left| \frac{m}{(m^2-1)^5} \right|$, converges.

Thus $\displaystyle\sum_{m=2}^{\infty} \frac{(-1)^m m}{(m^2-1)^5}$ **converges absolutely.**