1(b) If $f(x)$ does converge to 0 as $x \to \infty$, *must* $\int_a^\infty f(x)\,dx$ automatically converge to a finite number? That is, is $f(x) \to 0$ a *sufficient* condition for $\int_a^\infty f(x)\,dx$ to converge to a finite number?

Functions that converge to 0	What $\int_a^\infty f(x)\,dx$ does
xe^{-x^2}	converges
$\frac{1}{x^2}$	converges
$\frac{1}{x^3}$	converges
$\frac{1}{x}$	diverges

Thus knowing that $f(x) \to \infty$ is *not* sufficient information to conclude that $\int_a^\infty f(x)\,dx$ converges!

Important Lesson #1

- There is a huge distinction between $f(x)$ converging – that is, $\lim_{x\to\infty} f(x)$ being finite – and $I = \int_a^\infty f(x)\,dx$ converging. Just because you can find $\lim_{x\to\infty} f(x)$, and it's a finite number, does **not** mean that $\int_a^\infty f(x)\,dx$ will be finite.
- In fact, if $\lim_{x\to\infty} f(x)$ exists but is not 0, I diverges! No need to investigate any further.
- If $\lim_{x\to\infty} f(x) = 0$, I may converge or it may diverge – you must investigate further.

2. Let $I = \int_1^\infty \frac{1}{x^p}\,dx$. Determine whether I converges or diverges if

(a) $p > 1$:

$$\begin{aligned}\int_1^\infty \frac{1}{x^p}\,dx &= \lim_{R\to\infty}\int_1^R x^{-p}\,dx = \lim_{R\to\infty}\left.\frac{1}{-p+1}x^{-p+1}\right|_1^R \\ &= \lim_{R\to\infty}\frac{1}{1-p}R^{1-p} - \frac{1}{1-p}.\end{aligned}$$

$p > 1 \Rightarrow 1-p < 0 \Rightarrow p-1 > 0$

$$\lim_{R\to\infty} R^{1-p} = \lim_{R\to\infty}\frac{1}{R^{p-1}} = 0.$$

Thus $\int_1^\infty \frac{1}{x^p}\,dx = 0 - \frac{1}{1-p} = \frac{1}{p-1}$.

2(b) $p = 1$

We've already done this case. We know that $\displaystyle\int_1^\infty \frac{1}{x}\,dx = \infty$.

2(c) $0 < p < 1$:

As in the case of $p > 1$,

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{R\to\infty} \frac{1}{1-p}R^{1-p} - \frac{1}{1-p}.$$

$0 < p < 1 \Rightarrow 1 - p > 0 \Rightarrow \displaystyle\lim_{R\to\infty} R^{1-p} \to \infty$.

In this case, the integral again diverges.

2(d) $p = 0$:

$$\int_1^\infty \frac{1}{x^p}\, dx = \int_1^\infty 1\, dx = \infty,$$

as we've already seen.

2(e) $p < 0$:
Let $a = -p > 0$.

$$\int_1^\infty \frac{1}{x^p}\, dx = \int_1^\infty x^{-p}\, dx = \int_1^\infty x^a\, dx.$$

As $x \to \infty$, $x^a \to \infty$, and so the area under this curve is clearly infinite!