Determine whether each of the following improper integrals converges or diverges.

1. $\displaystyle\int_2^\infty \frac{1}{x^3+2}\,dx$

$$\begin{aligned} x^3+2 &> x^3 \\ 0 < \frac{1}{x^3+2} &< \frac{1}{x^3} \\ 0 \le \int_2^\infty \frac{1}{x^3+2}\,dx &\le \int_2^\infty \frac{1}{x^3}\,dx = \int_a^\infty \frac{1}{x^p}\,dx \text{ with } p>1 \end{aligned}$$

Since $p > 1$, $\displaystyle\int_2^\infty \frac{1}{x^3}\,dx$ converges (to $1/2$).

Thus $0 \le \displaystyle\int_2^\infty \frac{1}{x^3+2}\,dx \le$ a finite number, so it too must converge.

2. $\displaystyle\int_5^\infty \frac{1}{\sqrt{x}-2}\,dx$

$$\begin{aligned} 0 < \sqrt{x}-2 &< \sqrt{x} \text{ on } [5,\infty) \\ \frac{1}{\sqrt{x}-2} &> \frac{1}{\sqrt{x}} > 0 \text{ on } [5,\infty) \\ \int_5^\infty \frac{1}{\sqrt{x}-2}\,dx &\geq \int_5^\infty \frac{1}{\sqrt{x}}\,dx \geq 0 \end{aligned}$$

$\displaystyle\int_5^\infty \frac{1}{\sqrt{x}}\,dx = \int_5^\infty \frac{1}{x^p}\,dx$ with $p \leq 1$.

Since $p \leq 1$, $\displaystyle\int_5^\infty \frac{1}{\sqrt{x}}\,dx$ diverges to infinity.

Since $\displaystyle\int_5^\infty \frac{1}{\sqrt{x}}\,dx$ diverges (to infinity), and $\displaystyle\int_5^\infty \frac{1}{\sqrt{x}-2}\,dx$ is at least as large, it too must diverge.

3. $\displaystyle\int_2^\infty \frac{2}{\sqrt{x}+x^2}\,dx$

$$\begin{aligned}
\sqrt{x}+x^2 &> \sqrt{x} \text{ and } x^2 \geq 0 \\
0 \leq \frac{2}{\sqrt{x}+x^2} &< \frac{1}{\sqrt{x}} \text{ and } \frac{1}{x^2} \\
0 \leq \int_2^\infty \frac{2}{\sqrt{x}+x^2}\,dx &\leq \int_2^\infty \frac{1}{\sqrt{x}}\,dx \text{ and } \int_2^\infty \frac{1}{x^2}\,dx
\end{aligned}$$

$\displaystyle\int_2^\infty \frac{1}{\sqrt{x}}\,dx$ diverges to infinity, since $p = \frac{1}{2} < 1$.

Knowing that $\displaystyle\int_2^\infty \frac{2}{\sqrt{x}+x^2}\,dx \leq \infty$ is **not** helpful!

$\displaystyle\int_2^\infty \frac{1}{x^2}\,dx$ converges, since $p = 2 > 1$.

Thus $\displaystyle 0 \leq \int_2^\infty \frac{2}{\sqrt{x}+x^2}\,dx \leq$ a finite number, so it converges.

4. $\displaystyle\int_0^\infty \frac{2}{\sqrt{x}+x^2}\,dx$

Improper at both ends–split into 2 integrals.

Can split at any positive x-value - I choose $x = 2$.

$$\int_0^\infty \frac{2}{\sqrt{x}+x^2}\,dx = \int_0^2 \frac{2}{\sqrt{x}+x^2}\,dx + \int_2^\infty \frac{2}{\sqrt{x}+x^2}\,dx$$

We just found that $\displaystyle\int_2^\infty \frac{2}{\sqrt{x}+x^2}\,dx$ converges.

How about $\displaystyle\int_0^2 \frac{2}{\sqrt{x}+x^2}\,dx$? Using the same inequalities,

$$0 \le \int_0^1 \frac{2}{\sqrt{x}+x^2}\,dx \quad \le \quad \int_0^1 \frac{1}{\sqrt{x}}\,dx \text{ and } \int_0^1 \frac{1}{x^2}\,dx$$

In this case, the second integral on the right diverges, and so that's a true but useless comparison; but the first integral on the right converges, and so the original converges.

Putting it all together, we're adding up two finite pieces, and so the whole thing also converges.