Let $f(x) = e^x$.

(a) Find $P_1(x)$, $P_2(x)$, $P_3(x)$, and $P_4(x)$ for $f(x)$ based at $x_0 = 0$

$P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \ldots a_n(x - x_0)^n$, where $a_k = f^{(k)}(x_0)/k!$.

$f(x) = e^x$	$f(0) = 1$	$a_0 = 1/0! = 1$
$f'(x) = e^x$	$f'(0) = 1$	$a_1 = 1/1! = 1$
$f''(x) = e^x$	$f''(0) = 1$	$a_2 = 1/2! = 1/2$
$f^{(3)}(x) = e^x$	$f^{(3)}(0) = 1$	$a_3 = 1/3! = 1/6$
$f^{(4)}(x) = e^x$	$f^{(4)}(0) = 1$	$a_4 = 1/4! = 1/24$

Therefore,

$P_1(x) = 1 + x$

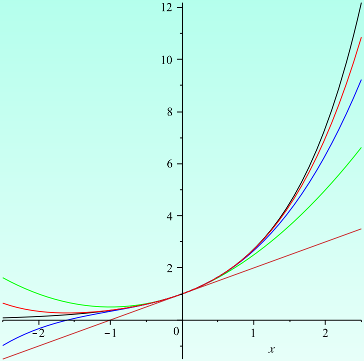
$P_2(x) = 1 + x + x^2/2$

$P_3(x) = 1 + x + x^2/2 + x^3/6$

$P_4(x) = 1 + x + x^2/2 + x^3/6 + x^4/24$

Notice: to get from $P_k(x)$ to $P_{k+1}(x)$, I just add the next term in, rather than starting all over again.

1(b) Graph e^x, $P_1(x)$, $P_2(x)$, $P_3(x)$ and $P_4(x)$ all on the same set of axes. Find intervals on which each Taylor polynomial is a good approximation.



Zooming in on the graphs ...

- P_1 is never particularly accurate ... maybe from about $[-0.01, 0.01]$ it's pretty good.
- P_2 is indistinguishable on about $[-.4, .4]$.
- P_3 is indistinguishable on about $[-.8, .8]$.
- P_4 is indistinguishable on about $[-1.5, 1.5]$

Moral: The higher degree polynomial you have, the farther away from your base point you can go and still have a good approximation.

1(c) Approximate $e^{1/2}$ using $P_4(x)$; compare to what Maple gives for $e^{1/2}$.

$$\begin{aligned} e^{-1/2} &\approx P_4(-\frac{1}{2}) \\ &\approx 1 - \frac{1}{2} + \frac{1}{2}\cdot\frac{1}{4} - \frac{1}{6}\cdot\frac{1}{8} + \frac{1}{24}\cdot\frac{1}{16} \\ &\approx \frac{233}{384} \approx 0.606771 \end{aligned}$$

Although it's hard to tell, it looks as if the graph of $P_4(x)$ is above the graph of e^x on the left of the y-axis, and so $P_4(-\frac{1}{2})$ is an over-estimate.
Maple gives 0.60653, so indeed P_4 did over-estimate at $x = -.5$.