Taylor's Theorem:

Let $f(x)$ be a function which is repeatedly differentiable on an interval I containing x_0. Suppose $P_n(x)$ is the n-th order Taylor polynomial based at x_0. Further suppose K_{n+1} is a bound for $|f^{(n+1)}(x)|$ on I. That is,

$$|f^{(n+1)}(x)| \leq K_{n+1} \text{ for all } x \in I$$

Then for all $x \in I$,

$$|f(x) - P_n(x)| \leq \frac{K_{n+1}}{(n+1)!}|x - x_0|^{n+1}$$

Let $f(x) = \sqrt{x}$.
Find $P_3(x)$ for $f(x)$ at the base point $x_0 = 64$.

$$f(x) = \sqrt{x} = x^{1/2} \qquad f(64) = 8 \qquad a_0 = \tfrac{f(64)}{0!} = 8$$

$$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}} \qquad f'(64) = \frac{1}{2\cdot 8} = \frac{1}{16} \qquad a_1 = f'(64)/1! = \frac{1}{16}$$

$$f''(x) = -\frac{1}{4}x^{-3/2} \qquad f''(64) = -1/2048 \qquad a_2 = f''(64)/2! = -\tfrac{1}{4096}$$

$$f'''(x) = \frac{3}{8}x^{-5/2} \qquad f'''(64) = 3/262144 \qquad a_3 = f'''(64)/3! = \tfrac{1}{524288}$$

$$f^{(4)}(x) = -\frac{15}{16}x^{-7/2}$$

Thus

$$P_3(x) = 8 + \frac{1}{16}(x-64) - \frac{1}{4096}(x-64)^2 + \frac{1}{524288}(x-64)^3.$$

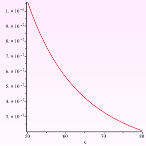
What can you say about the error committed by using $P_3(x)$ as an approximation for $\sqrt{x}$ on the interval $[50, 80]$?

Taylor's theorem:

$$|f(x) - P_3(x)| \leq \frac{K_4}{(4)!}|x - 64|^4,$$

Need to choose $K_4 \geq |f^{(4)}(x)|$ on $[50, 80]$
$|f^{(4)}(x)| = \frac{15}{16}x^{-7/2}$.
Looking at the graph of $|f^{(4)}(x)|$ on $[50, 80]$, can choose $K_4 = 0.0000012$.

$$|\sqrt{x} - P_3(x)| \leq \frac{.0000012}{4!}|80 - 64|^4 = .00328.$$



Approximating π

Notice that $\arctan(1) = y \Leftrightarrow \tan(y) = 1 \Leftrightarrow \sin(y) = \cos(y)$, so

$$\arctan(1) = \pi/4.$$

Plan: Approximate π by finding a Taylor polynomial for $\arctan(x)$ based at $x = 0$ and using it to approximate $\arctan(1) = \pi/4$.

$f(x) = \arctan(x)$	$f(0) = 0$	$a_0 = \frac{0}{0!} = 0$
$f'(x) = \frac{1}{1+x^2}$	$f'(0) = 1$	$a_1 = \frac{1}{1!}! = 1$
$f''(x) = -\frac{2x}{(1+x^2)^2}$	$f''(0) = 0$	$a_2 = \frac{0}{2!} = 0$
$f^{(3)}(x) = \frac{8x^2}{(1+x^2)^3} - \frac{2}{(1+x^2)^2}$	$f^{(3)}(0) = -2$	$a_3 = -\frac{2}{3!} = -\frac{1}{3}$
$f^{(4)}(x) = \frac{-48x^3}{(1+x^2)^4} + \frac{24x}{(1+x^2)^3}$	$f^{(4)}(0) = 0$	$a_4 = \frac{0}{4!} = 0$
$f^{(5)}(x) = \frac{384x^4}{(1+x^2)^5} - \frac{288x^2}{(1+x^2)^4} + \frac{24}{(1+x^2)^3}$	$f^{(5)}(0) = 24 = 4!$	$a_5 = \frac{4!}{5!} = \frac{1}{5}$
	$f^{(6)}(0) = 0$	$a_6 = 0$
	$f^{(7)}(0) = -6!$	$a_7 = -\frac{6!}{7!} = -\frac{1}{7}$

Based on these results,

$$P_7(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7},$$

and in fact, building on the pattern we see here

$$P_{99}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + \frac{x^{97}}{97} - \frac{x^{99}}{99}.$$

Thus

$$\begin{aligned} \frac{\pi}{4} = \arctan(1) &\approx 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{1}{97} - \frac{1}{99} \\ &\approx 0.7803986631 \\ \Rightarrow \pi &\approx 3.121594652 \end{aligned}$$