Do the following sequences converge or diverge?
If the sequence converges, find the limit.

1. $\left\{ \dfrac{\sqrt{n}}{\ln(n)} \right\}_{n=2}^{\infty}$

First couple terms:

$$\left\{ \frac{\sqrt{2}}{\ln(2)}, \frac{\sqrt{3}}{\ln(3)}, \frac{\sqrt{4}}{\ln(4)}, \dots \right\} \approx \{2.04, 1.58, 1.44, \dots\}$$

Does it converge or diverge?

$$\begin{aligned}
\lim_{n\to\infty} \frac{\sqrt{n}}{\ln(n)} &= \frac{\infty}{\infty}. \text{ Indeterminate form! Use l'Hôpital's Rule} \\
&= \lim_{n\to\infty} \frac{\frac{1}{2\sqrt{n}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{2\sqrt{n}} = \lim_{n\to\infty} \frac{\sqrt{n}}{2} \\
&= \infty
\end{aligned}$$

Thus this sequence diverges (to ∞).

2. $\left\{ \dfrac{100m^2 + 200}{.01m^3 - 500m^2} \right\}_{m=2}^{\infty}$

First couple terms:

$$\left\{ \frac{600}{-1999.92}, \frac{1100}{-4499.73}, \frac{1800}{-7999.36}, \dots \right\} \approx \{-.300, -.244, -.225, \dots\}$$

$$\begin{aligned}
\lim_{m\to\infty} \frac{100m^2 + 200}{.01m^3 - 500m^2} &= \frac{\infty}{\infty}. \text{ Indeterminate form! Use l'Hôpital's} \\
&= \lim_{m\to\infty} \frac{200m}{.03m^2 - 1000m} = \frac{\infty}{\infty}. \text{ Use l'Hôpital's} \\
&= \lim_{m\to\infty} \frac{200}{.03m - 1000} = \frac{200}{\infty} \\
& \quad \textbf{NOT} \text{ indeterminate} \\
&= 0
\end{aligned}$$

This sequence converges, to 0.

3. $\left\{ \dfrac{5e^{6n} + 100n}{10e^{6n} + n^{100}} \right\}_{n=0}^{\infty}$

First couple terms:

$$\left\{ \frac{5}{10}, \frac{5e^6 + 100}{10e^6 + 1}, \frac{5e^{12} + 200}{10e^{12} + 2^{100}}, \ldots \right\} \approx \{0.5, 0.525, 6.42 * 10^{-25}, 6.37 * 10^{-40}, 8.24 * 10^{-50}, 6.77 * 10^{-57}, \ldots\}$$

Looks as if it's going to approach 0!

Except... The leaps in order of magnitude are getting smaller, so maybe it's slowing down. Maybe it will approach some very very very small non-zero number?
Let's see!

3. (continued)

$$
\begin{aligned}
\lim_{n\to\infty} \frac{5e^{6n}+100n}{10e^{6n}+n^{100}} &= \frac{\infty}{\infty} \text{ l'Hôpital's!} \\
&= \lim_{n\to\infty} \frac{6\cdot 5e^{6n}+100}{6\cdot 10e^{6n}+100n^{99}} = \frac{\infty}{\infty} \text{ l'Hôpital's} \\
&= \lim_{n\to\infty} \frac{6\cdot 6\cdot 5e^{6n}}{6\cdot 6\cdot 10e^{6n}+100\cdot 99n^{98}} = \frac{\infty}{\infty} \\
&\vdots \\
&= \lim_{n\to\infty} \frac{6^{100}5e^{6n}}{6^{100}10e^{6n}+100!} = \frac{\infty}{\infty} \\
&= \lim_{n\to\infty} \frac{6^{101}5e^{6n}}{6^{101}10e^{6n}} \\
&= \frac{1}{2}
\end{aligned}
$$

Thus this sequence converges, but not to 0, or to a very small non-zero number. It converges to $\frac{1}{2}$ – its starting term!

4. $\{j\sin(1/j)\}_{j=10}^{\infty}$

First couple terms:

$$\left\{10\sin\left(\frac{1}{10}\right), 11\sin\left(\frac{1}{11}\right), 12\sin\left(\frac{1}{12}\right), \dots\right\} \approx \{0.9983, 0.9986, 0.9988, \dots\}$$

$$\begin{aligned}\lim_{j\to\infty} j\sin(1/j) &\qquad \text{Since } 1/j \to 0, \text{ and as } x \to 0, \sin(x) \to 0 \\ &\qquad \text{we conclude that } \sin(1/j) \to 0 \\ &= \infty \cdot 0 \text{ Indeterminate!}\end{aligned}$$

To use l'Hôpital's Rule, our general term needs to be in the form $\frac{f(j)}{g(j)}$.

$$\begin{aligned}\lim_{j\to\infty} j\sin(1/j) &= \lim_{j\to\infty} \frac{\sin(1/j)}{1/j} = \lim_{j\to\infty} \frac{\cos(1/j)\cdot -1/j^2}{-1/j^2} \\ &= \lim_{j\to\infty} \cos(1/j) \\ &= 1\end{aligned}$$

Thus this sequence converges, to 1.

5. $\left\{ \dfrac{\sin(k)}{k^2} \right\}_{k=1}^{\infty}$

First couple terms:

$$\left\{ \frac{\sin(1)}{1}, \frac{\sin(2)}{4}, \frac{\sin(3)}{9}, \frac{\sin(4)}{16}, \dots \right\}$$

L'Hôpital's does not apply, because while the denominator approaches ∞, the numerator is just ranges along values between -1 and 1, and so our limit is not in indeterminate form.

Even though the limit of the numerator does not exist, because it's bounded while the denominator is increasing without bound, this series converges, to 0.