Definition:

If f is continuous on the interval $[a, b)$, but $|f(x)| \to \infty$ as $x \to b^-$, we define the **improper integral** of f on $[a, b]$ by

$$\int_a^b f(x)\,dx = \lim_{R \to b^-} \int_a^R f(x)\,dx.$$

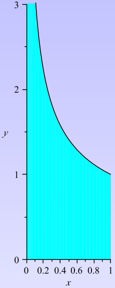
Similarly, if f is continuous on the interval $(a, b]$, but $|f(x)| \to \infty$ as $x \to a^+$, we define the improper integral

$$\int_a^b f(x)\,dx = \lim_{R \to a^+} \int_R^b f(x)\,dx.$$

In either case, if the limit exists and equals some finite value L, we say that the improper integral **converges** to L. If the limit does not exist (whether because it equals $\pm\infty$ or because for some other reason), we say that the improper integral **diverges**.

1. Determine whether the following improper integrals converge or diverge. For those that converge, what value do they converge to?
   1.1 $\int_0^1 \frac{1}{x^{1/3}}\,dx$
   1.2 $\int_0^{\pi/2} \frac{\sin(x)}{\sqrt{\cos(x)}}\,dx$
   1.3 $\int_0^1 \frac{1}{x}\,dx$
   1.4 $\int_0^1 \frac{1}{x^3}\,dx$
2. For what values of p is $\int_0^1 \frac{1}{x^p}\,dx$ improper? For which of those values does the integral converge?
3. First determine why $\int_0^1 x\ln(x)\,dx$ is improper, and then (try to) evaluate it. Is there a difficulty?

Consider again the improper integral $\int_0^1 \frac{1}{\sqrt{x}}\,dx$. We've seen that this converges, to 2.



This "area" is 2, whether we find it by integrating with respect to x or with respect to y! In other words ...

$$2 = \int_0^1 \frac{1}{\sqrt{x}}\,dx = \int_0^1 1\,dy + \int_1^\infty \frac{1}{y^2}\,dy.$$

Definition:

If $f(x)$ is continuous on the interval $[a, \infty)$, we define the **improper integral** $\int_a^\infty f(x)\,dx$ to be

$$\int_a^\infty f(x)\,dx \overset{\text{def}}{=} \lim_{R\to\infty} \int_a^R f(x)\,dx.$$

Similarly, if $f(x)$ is continuous on the interval $(-\infty, a]$, we define

$$\int_{-\infty}^a f(x)\,dx \overset{\text{def}}{=} \lim_{R\to\infty} \int_{-R}^a f(x)\,dx.$$

In either case, if the limit exists (and equals some value L), we say that the improper integral **converges** (to L). If the limit does not exist (whether because it is infinite or for other reasons), we say that the improper integral **diverges**.

1. Each of the following integrals is improper because the interval of integration is infinite. Determine whether each integral converges or diverges, and if it does converge, what it converges to.

   a. $\displaystyle\int_1^\infty \frac{1}{x^3}\ dx$

   b. $\displaystyle\int_1^\infty \frac{1}{x}\ dx$

   c. $\displaystyle\int_1^\infty 1+\frac{1}{x^2}\ dx$

   d. $\displaystyle\int_1^\infty \frac{1}{x^p}\ dx$ where $p>1$

2. Think about the above results and the big picture of what's going on.

   2.1 Is it *necessary* that $f(x)$ converge to 0 as $x \to \infty$ in order for $\displaystyle\int_a^\infty f(x)\,dx$ to converge to a finite number?

   2.2 If $f(x)$ does converge to 0 as $x \to \infty$, *must* $\displaystyle\int_a^b f(x)\,dx$ automatically converge to a finite number? That is, is $f(x) \to 0$ a *sufficient* condition for $\displaystyle\int_a^\infty f(x)\,dx$ to converge to a finite number?