1. Determine whether the following improper integrals converge or diverge. For those that converge, what value do they converge to?

   (a) $\int_0^1 \frac{1}{x^{1/3}}\,dx$

$$\begin{aligned}\int_0^1 \frac{1}{x^{1/3}}\,dx &= \lim_{R\to 0^+} \int_R^1 x^{-1/3}\,dx \\ &= \lim_{R\to 0^+} \frac{3}{2}x^{2/3}\Big|_R^1 \\ &= \frac{3}{2}\lim_{R\to 0^+} 1 - R^{2/3} \\ &= \frac{3}{2}\end{aligned}$$

Thus the improper integral converges, to $\frac{3}{2}$.

1(b) $\displaystyle\int_0^{\pi/2} \frac{\sin(x)}{\sqrt{\cos(x)}}\,dx$

Let $u = \cos(x)$, so $du = -\sin(x)\,dx$.

$x = 0 \Rightarrow u = 1$, $x = \pi/2 \Rightarrow u = 0$.

$$\begin{aligned}
\int_0^{\pi/2} \frac{\sin(x)}{\sqrt{\cos(x)}}\,dx &= -\int_1^0 \frac{1}{\sqrt{u}}\,du \\
&= \lim_{R\to 0^+} \int_R^1 u^{-1/2}\,du \\
&= \lim_{R\to 0^+} 2\sqrt{u}\Big|_R^1 \\
&= \lim_{R\to 0^+} 2 - 2\sqrt{R} \\
&= 2
\end{aligned}$$

Thus this improper integral converges, to 2.

1(c) $\int_0^1 \frac{1}{x}\,dx$

$$\begin{aligned}\int_0^1 \frac{1}{x}\,dx &= \lim_{R\to 0^+}\int_R^1 \frac{1}{x}\,dx\\ &= \lim_{R\to 0^+} \ln(x)\Big|_R^1\\ &= \lim_{R\to 0^+} 0-\ln(R)\\ &= -(-\infty)=\infty\end{aligned}$$

Thus this improper integral diverges (to infinity).

1(d) $\int_0^1 \frac{1}{x^3}\,dx$

$$\begin{aligned}\int_0^1 \frac{1}{x^3}\,dx &= \lim_{R\to 0^+} \int_R^1 x^{-3}\,dx \\ &= \lim_{R\to 0^+} -\frac{1}{2}x^{-2}\Big|_R^1 \\ &= \lim_{R\to 0^+} -\frac{1}{2} + \frac{1}{2R^2} \\ &= \infty\end{aligned}$$

This improper integral also diverges (to infinity).

2. For what values of p is $\int_0^1 \frac{1}{x^p}\,dx$ improper? For which of those values does the integral converge?

- If $p < 0$, we have $\int_0^1 \frac{1}{x^{\text{negative power}}}\,dx = \int_0^1 x^{\text{positive power}}\,dx$, which is not improper.
- If $p = 0$, we have $\int_0^1 1\,dx$, which is not improper.
- When $p > 0$, we really do have a positive power of x in the denominator, so at $x = 0$ we really do have a problem.

Thus $\int_0^1 \frac{1}{x^p}\,dx$ is improper only for $p > 0$.

1. (continued)

For which positive values of p does $\int_0^1 \frac{1}{x^p}\,dx$ converge?

$$\begin{aligned}\int_0^1 \frac{1}{x^p}\,dx &= \lim_{R\to 0^+}\int_R^1 x^{-p}\,dx = \lim_{R\to 0^+} -\frac{1}{-p+1}x^{-p+1}\Big|_R^1 \\ &= \lim_{R\to 0^+}\frac{1}{p-1}x^{1-p}\Big|_R^1 = \lim_{R\to 0^+}\frac{1}{p-1} - \frac{1}{p-1}R^{1-p}\end{aligned}$$

If $1-p>0$, $\int_0^1 \frac{1}{x^p}\,dx = \frac{1}{p-1} + 0 \Rightarrow$ **converges** if $0<p<1$

If $1-p<0$, then $p-1>0$ and

$$\int_0^1 \frac{1}{x^p}\,dx = \frac{1}{p-1} - \lim_{R\to 0^+}\frac{1}{p-1}\frac{1}{R^{p-1}} = \infty$$

Thus if $p>1$, the improper integral **diverges.**

3. $\int_0^1 x \ln(x)\, dx = \lim_{R \to 0^+} \int_R^1 x \ln(x)\, dx.$

Let

$u = \ln(x) \qquad dv = x\, dx$

$du = \frac{1}{x}\, dx \qquad v = \frac{1}{2}x^2$

$$
\begin{aligned}
\int_0^1 x \ln(x)\, dx &= \lim_{R \to 0^+} \left[\frac{x^2}{2} \ln(x) - \int \frac{x^2}{2x}\, dx \right]_R^1 \\
&= \lim_{R \to 0^+} \left[\frac{x^2}{2} \ln(x) - \frac{1}{4}x^2 \right]_R^1 = \lim_{R \to 0^+} \frac{x^2}{4} \Big[2\ln(x) - 1 \Big]_R^1 \\
&= \lim_{R \to 0^+} \left[\frac{1}{4}(-1) - \frac{R^2}{4}\big(2\ln(R) - 1\big) \right] \\
&= -\frac{1}{4} - \lim_{R \to 0^+} \frac{R^2}{2}\big(\ln(R)\big) + \lim_{R \to 0^+} \frac{R^2}{4} \\
&= -\frac{1}{4} - \lim_{R \to 0^+} \frac{R^2}{2} \cdot \lim_{R \to 0^+} \ln(R) + 0 \\
&= \ ?
\end{aligned}
$$