1(a) $\int_1^\infty \frac{1}{x^3}\ dx$

- $1/x^3$ converges to 0 as $x \to \infty$
-

$$\begin{aligned} \int_1^\infty \frac{1}{x^3}\ dx &= \lim_{R\to\infty} \int_1^R x^{-3}\ dx = \lim_{R\to\infty} (-\frac{1}{2} \cdot \frac{1}{x^2})\Big|_1^R \\ &= \lim_{R\to\infty} (-\frac{1}{2R^2} + \frac{1}{2}) = \frac{1}{2} \end{aligned}$$

This improper integral converges, to $\frac{1}{2}$.

1(b) $\int_1^\infty 1 + \frac{1}{x^2}\,dx$

- $1 + \frac{1}{x^2} \to 1$ as $x \to \infty$
- $$\int_1^\infty 1 + \frac{1}{x^2}\,dx = \int_1^\infty 1\,dx + \int_1^\infty \frac{1}{x^2}\,dx$$

  We've seen that the first integral on the right diverges (to ∞), the second one converges (to 1).

  Because this sum does not approach a finite number, it *diverges*.

Notice: f converges as $x \to \infty$ but $\int_a^\infty f$ diverges.

1(c) $\int_1^{\infty} \frac{1}{x}\, dx$

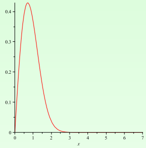
- $\frac{1}{x} \to 0$ as $x \to \infty$
- $$\begin{aligned} \int_1^{\infty} \frac{1}{x}\, dx &= \lim_{R\to\infty} \int_1^{R} \frac{1}{x}\, dx = \lim_{R\to\infty} \ln(x)\Big|_1^R \\ &= \lim_{R\to\infty} (\ln(R) - \ln(1)) = \lim_{R\to\infty} \ln(R) = \infty \end{aligned}$$

  This improper integral diverges (slowly).

Notice: f converges to 0 as $x \to \infty$ but $\int_a^{\infty} f$ diverges!

1(d) $\int_0^{\infty} xe^{-x^2}\, dx$

- As $x \to \infty$, $\frac{x}{e^{x^2}} \to \frac{\infty}{\infty}$. Another limit we can't do b/c it's in *indeterminate form*! Looking at a graph of xe^{-x^2}, can see that integrand approaches 0.



- Let $u = -x^2$, so $du = -2x\, dx$, or $-\frac{1}{2}\, du = x\, dx$
  Also, $x = 0 \Rightarrow u = 0$; $x = \infty \Rightarrow u = -\infty$.

$$\begin{aligned}\int_0^{\infty} xe^{-x^2}\, dx &= -\frac{1}{2}\int_0^{-\infty} e^u\, du = \frac{1}{2}\int_{-\infty}^{0} e^u\, du = \frac{1}{2}\lim_{R\to\infty} e^u\Big|_{-R}^{0} \\ &= \frac{1}{2}\lim_{R\to\infty} 1 - e^{-R} = 1 - \lim_{R\to\infty}\frac{1}{e^R} = 1\end{aligned}$$

The improper integral converges, to 1.

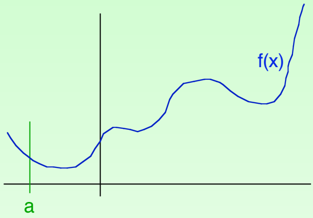
2. Think about all the results you've seen, as well as the big picture.

(a) Is it *necessary* that $f(x)$ converge to 0 as $x \to \infty$ in order for $\int_a^\infty f(x)\,dx$ to converge to a finite number?

Convergent integral	what f converges to
$\int_1^\infty \frac{1}{x^2}\,dx$	$\frac{1}{x^2} \to 0$
$\int_1^\infty \frac{1}{x^3}\,dx$	$\frac{1}{x^3} \to 0$
$\int_0^\infty xe^{-x^2}\,dx$	$xe^{-x^2} \to 0$

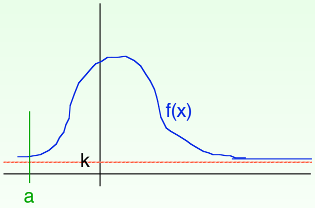
For what it's worth, so far every example that we've seen of a convergent improper integral *has* at an integrand that converges to 0 as $x \to \infty$. But that's not enough.

2(a) (continued)



As $x \to \infty$, $f(x) \to \infty$,

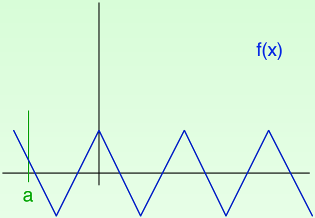
$$\text{and} \int_a^\infty f(x)\,dx = \infty$$



As $x \to \infty$, $f(x) \to k \neq 0$, and

$$\begin{aligned}\int_a^\infty f(x)\,dx &= \lim_{R\to\infty}\int_a^R f(x)\,dx \\ &> \lim_{R\to\infty}\int_a^R k\,dx \\ &> \lim_{R\to\infty} kR = \pm\infty\end{aligned}$$

2(a) (continued)



As $x \to \infty$, $\lim_{x\to\infty} f(x)$ d.n.e., and $\int_a^\infty f(x)\,dx = \lim_{R\to\infty} f(x)\,dx$ d.ne., so the integral diverges

Conclusion: The only way $\int_a^\infty f(x)\,dx$ can have a **hope** of converging to a finite number is if $\lim_{x\to\infty} f(x) = 0$.

In other words, if $\lim_{x\to\infty} f(x) \neq 0$, $\int_a^\infty f(x)\,dx$ **must** diverge.

2(b) If $f(x)$ does converge to 0 as $x \to \infty$, *must* $\int_a^b f(x)\,dx$ automatically converge to a finite number? That is, is $f(x) \to 0$ a *sufficient* condition for $\int_a^\infty f(x)\,dx$ to converge to a finite number?

Functions that converge to 0	What $\int_a^\infty f(x)\,dx$ does
xe^{-x^2}	converges
$\frac{1}{x^2}$	converges
$\frac{1}{x^3}$	converges
$\frac{1}{x}$	diverges

Thus knowing that $f(x) \to \infty$ is *not* sufficient information to conclude that $\int_a^\infty f(x)\,dx$ converges!