1. Find the following derivatives. Don't worry about algebraic simplifications.

   (a) $\dfrac{d}{dx}\left(\arcsin\left(\dfrac{2x}{e^x}\right)\right)$

   $\dfrac{2x}{e^x}$ is *inside* $\arcsin(x) \Longrightarrow$ use the *chain rule*.

$$\begin{aligned}
\frac{d}{dx}\left(\arcsin\left(\frac{2x}{e^x}\right)\right) &= \frac{1}{\sqrt{1-\left(\frac{2x}{e^x}\right)^2}} \cdot \frac{d}{dx}\left(\frac{2x}{e^x}\right) \\
&= \frac{1}{\sqrt{1-\left(\frac{2x}{e^x}\right)^2}} \cdot \frac{e^x \cdot 2 - 2x \cdot e^x}{(e^x)^2} \\
&= \frac{1}{\sqrt{1-\left(\frac{2x}{e^x}\right)^2}} \cdot \frac{2-2x}{e^x}
\end{aligned}$$

(b) $\dfrac{d}{dx}(x^2 \arctan(\ln(x))$

product of x^2 and $\arctan(\ln(x)) \Longrightarrow$ begin with product rule

$$\begin{aligned}
\frac{d}{dx}(x^2 \arctan(\ln(x)) &= x^2 \cdot \frac{d}{dx}(\arctan(\ln(x)) + \frac{d}{dx}(x^2) \cdot \arctan(\ln(x)) \\
\textit{(now use chain rule!)} \quad &= x^2 \cdot \frac{1}{1+(\ln(x))^2} \cdot \frac{d}{dx}(\ln(x)) + 2x \cdot \arctan(\ln(x) \\
&= x^2 \cdot \frac{1}{1+(\ln(x))^2} \cdot \frac{1}{x} + 2x \arctan(\ln(x)) \\
&= \frac{x}{1+(\ln(x))^2} + 2x \arctan(\ln(x))
\end{aligned}$$

(c) $\dfrac{d}{dx}(\arctan(\arcsin(x)))$

Chain rule!

$$\begin{aligned}
\frac{d}{dx}(\arctan(\arcsin(x))) &= \frac{1}{1+(\arcsin(x))^2} \cdot \frac{d \arcsin(x)}{dx} \\
&= \frac{1}{1+(\arcsin(x))^2} \cdot \frac{1}{\sqrt{1-x^2}} \\
&= \frac{1}{(1+(\arcsin(x))^2)\sqrt{1-x^2}}
\end{aligned}$$

2. For each of the following, find an antiderivative.

(a) $g(x) = \dfrac{3}{x^2}$
Rewrite $g(x)$ as $3x^{-2}$:

$$G(x) = 3 \cdot \frac{x^{-1}}{-1} = -\frac{3}{x}.$$

(b) $h(x) = \dfrac{3}{1+x^2}$
This I recognize as being related to the $\arctan(x)$.

$$h(x) = 3 \cdot \frac{1}{1+x^2} \Longrightarrow H(x) = 3 \cdot \arctan(x).$$

(c) $k(x) = \dfrac{7}{\sqrt{1-(x/6)^2}}$

Guess and check!

- $\sqrt{1-\square^2}$ in denominator
- Just a constant in the numerator
- $\Longrightarrow$ antiderivative probably has something to do with arcsin
- For any u, $7 \cdot \dfrac{d}{du}(\arcsin(u)) = \dfrac{7}{\sqrt{1-u^2}}$
- $\dfrac{7}{\sqrt{1-(x/6)^2}}$ must have come from $7\arcsin(x/6)$ in some way.
- In $\dfrac{d}{dx}\left(\dfrac{7}{\sqrt{1-(x/6)^2}}\right)$, the chain rule would give an extra factor of $1/6$.
- Compensate by dividing by $1/6$ (or multiplying by 6).

$$k(x) = 7 \cdot \frac{1}{\sqrt{1-(x/6)^2}} \Longrightarrow K(x) = 7 \cdot 6 \cdot \arcsin(x/6).$$

Check: $K'(x) = 7 \cdot 6 \cdot \dfrac{1}{\sqrt{1-(x/6)^2}} \cdot \dfrac{1}{6} = \dfrac{7}{\sqrt{1-(x/6)^2}} = k(x)$

3. For each of the following, find the signed area given by the integral shown.

(a) $\displaystyle\int_0^1 \frac{1}{4\sqrt{1-x^2}}\,dx$

FTC Part 1 $\Longrightarrow$ antidiff. the integrand, plug in limits of integration:

$$\begin{aligned}
\int_0^1 \frac{1}{4\sqrt{1-x^2}}\,dx &= \frac{1}{4}\arcsin(x)\Big|_0^1 \\
&= \frac{1}{4}(\arcsin(1)-\arcsin(0)) \\
&= \frac{1}{4}(\pi/2-0) \\
&= \pi/8
\end{aligned}$$

(b) $\displaystyle\int_{-\pi/2}^{\pi/3} \cos(-3x)\, dx$

As in 2(c), compensate for the factor of -3 in the cosine function by dividing by -3 outside:

$$\begin{aligned}
\int_{-\pi/2}^{\pi/3} \cos(-3x)\, dx &= -\frac{1}{3}\sin(-3x)\Big|_{-\pi/2}^{\pi/3} \\
&= -\frac{1}{3}(\sin(-\pi) - \sin(3\pi/2)) \\
&= -\frac{1}{3}(0 - (-1)) \\
&= -\frac{1}{3}
\end{aligned}$$

(c) $\displaystyle\int_0^1 \frac{1}{1+9x^2}\,dx$

$$\frac{1}{1+9x^2} = \frac{1}{1+(3x)^2} \Rightarrow \text{similar to 2(c).}$$

$$\begin{aligned}\int_0^1 \frac{1}{1+(3x)^2}\,dx &= \frac{1}{3}\arctan(3x)\Big|_0^1 = \frac{1}{3}(\arctan(3) - \arctan(0)) \\ &= \frac{\arctan(3)}{3}\end{aligned}$$

(d) $\displaystyle\int_1^{e^{\pi/4}} \frac{\sin(\ln(x))}{x}\,dx$

- Much harder to do with guess and check!
- Need to think of something that differentiates to $\dfrac{\sin(\ln(x))}{x}$.
- $\ln(x)$ inside sin $\Rightarrow$ integrand must come from differentiating using the chain rule.
- Part of chain rule: differentiate outer fn leaving inner fn the same $\Rightarrow$ part of original fn must be $-\cos(\ln(x))$.

Check to see what differentiating this gives us, so we can get a feel for other necessary adjustments:

$$\begin{aligned}\frac{d}{dx}(-\cos(\ln(x))) &= \sin(\ln(x))\cdot\frac{d}{dx}(\ln(x))\\ &= \sin(\ln(x))\cdot\frac{1}{x} = \frac{\sin(\ln(x))}{x}.\end{aligned}$$

Lo and behold, we have our antiderivative.

$$\int_1^{e^{\pi/4}} \frac{\sin(\ln(x))}{x}\,dx = -\cos(\ln(x))\Big|_1^{e^{\pi/4}} = -\frac{\sqrt{2}}{2}+1.$$

If you find this guessing and checking method very hard to follow, you are *not* alone. It was to surely avoid just such pain and agony that various methods of integration were developed. We'll learn two in this class – substitution and integration by parts.