1. $\displaystyle\int \frac{1}{\sqrt{1-x}}\,dx \qquad (u = 1 - x)$

- Substitute:
  - Composition: $\sqrt{1-x}$.
  - Let $u = 1 - x$.
  - Differentiating $u \Rightarrow \dfrac{du}{dx} = -1 \Rightarrow du = -1\,dx \Rightarrow dx = -1\,du$.
  - Replacing $1 - x$ by u and dx by $-1\,du$ in the original integral:

$$\int \frac{1}{\sqrt{1-x}}\,dx = \int \frac{1}{\sqrt{u}} \cdot -1\,du = -\int u^{-1/2}\,du.$$

- Antidifferentiate in u:

$$-\int u^{-1/2}\,du = -\frac{1}{1/2}u^{1/2} + C = -2\sqrt{u} + C.$$

- Resubstitute:

$$\int \frac{1}{\sqrt{1-x}}\,dx = -2\sqrt{1-x} + C.$$

Check:

$$\frac{d}{dx}\left(-2\sqrt{1-x} + C\right) = -2 \cdot \frac{1}{2}(1-x)^{-1/2} \cdot (-1) + 0 = \frac{1}{\sqrt{1-x}}$$

2. $\int x \sin(\pi x^2)\, dx \qquad (u = \pi x^2)$

- Substitute:
  - Composition: $\sin(\pi x^2)$
  - Let $u = \pi x^2$.
  - Differentiating $u \Rightarrow \frac{du}{dx} = 2\pi x \Rightarrow du = 2\pi x\, dx \Rightarrow x\, dx = \frac{1}{2\pi}\, du$.
  - Replacing πx^2 with u and $x\, dx$ with $\frac{1}{2\pi}\, du$:

$$\int x \sin(\pi x^2)\, dx = \int \sin(\pi x^2) \cdot x\, dx = \int \sin(u) \cdot \frac{1}{2\pi}\, du = \frac{1}{2\pi} \int \sin(u)\, du.$$

- Antidifferentiate in u:

$$\frac{1}{2\pi} \int \sin(u)\, du = \frac{1}{2\pi} \cdot -\cos(u) + C.$$

- Resubstitute:

$$\int x \sin(\pi x^2)\, dx = -\frac{1}{2\pi} \cos(\pi x^2) + C.$$

Check:

$$\frac{d}{dx}\left(-\frac{1}{2\pi}\cos(\pi x^2) + C\right) = -\frac{1}{2\pi} \cdot -sin(\pi x^2) \cdot 2\pi x = x \sin(\pi x^2)$$

3. $\displaystyle\int_1^3 \frac{x}{1+x^2}\,dx \qquad (u = 1+x^2)$

- Substitute:
  - Composition: $\dfrac{x}{1+x^2} = x(1+x^2)^{-1}$.
  - Let $u = 1 + x^2$
  - Differentiating $u \Rightarrow \dfrac{du}{dx} = 2x \Rightarrow du = 2x\,dx \Rightarrow x\,dx = \dfrac{1}{2}\,du$.
  - Replacing $1+x^2$ with u and $x\,dx$ with $\frac{1}{2}\,du$:

$$\int_1^3 \frac{x}{1+x^2}\,dx = \int_{x=1}^{x=3} \frac{1}{1+x^2}\cdot x\,dx = \int_{u=2}^{u=10} \frac{1}{u}\cdot\frac{1}{2}\,du = \frac{1}{2}\int_2^{10}\frac{1}{u}\,du.$$

- Antidifferentiate in u:

$$\frac{1}{2}\int_2^{10}\frac{1}{u}\,du = \frac{1}{2}\ln|u|\Big|_{10}^{2} = \frac{1}{2}\left(\ln|10| - \ln|2|\right) = \frac{1}{2}\ln\left(\frac{10}{2}\right) = \ln(5^{1/2})$$

Check:

$$\frac{d}{dx}\left(\frac{1}{2}\ln|1+x^2|\right) = \frac{1}{2}\cdot\frac{1}{1+x^2}\cdot 2x = \frac{x}{1+x^2}.$$

4. $\int \frac{x}{1+x^4}\,dx \qquad (u = x^2)$

- Substitute:
  - Composition: $\frac{x}{1+x^4} = x(1+x^4)^{-1} = x(1+(x^2)^2)^{-1}$.
  - Letting $u = x^4$ won't work. In that case, $du = 4x^3\,dx$, and there is no $x^3\,dx$ term present.
  - Instead, let $u = x^2$.
  - Differentiating $u \Rightarrow \frac{du}{dx} = 2x \Rightarrow du = 2x\,dx \Rightarrow x\,dx = \frac{1}{2}\,du$.
  - Replacing x^2 with u and $x\,dx$ with $\frac{1}{2}\,du$:

$$\int \frac{x}{1+x^4}\,dx = \int \frac{1}{1+(x^2)^2} \cdot x\,dx = \int \frac{1}{1+u^2} \cdot \frac{1}{2}\,du = \frac{1}{2}\int \frac{1}{1+u^2}\,du.$$

- Antidifferentiate in u:

$$\frac{1}{2}\int \frac{1}{1+u^2}\,du = \frac{1}{2}\arctan(u) + C.$$

- Resubstitute:

$$\int \frac{x}{1+x^4}\,dx = \frac{1}{2}\arctan(x^2) + C.$$

Remember to check by differentiating!

5. $\displaystyle\int_2^5 \frac{1}{x\ln(x)}\,dx \qquad (u = \ln(x))$

- Substitute:
  - Composition: $(x\ln(x))^{-1} = \frac{1}{x}(\ln(x))^{-1}$.
  - Letting $u = x\ln(x)$ results in $du = \big(1 + \ln(x)\big)\,dx$. We don't have anything related to du present.
  - Let $u = \ln(x)$.
  - Differentiating $u \Rightarrow \dfrac{du}{dx} = \dfrac{1}{x} \Rightarrow du = \dfrac{1}{x}\,dx$.
  - Replacing $\ln(x)$ with u and $\frac{1}{x}\,dx$ with du:

$$\int_2^5 \frac{1}{x\ln(x)}\,dx = \int_{x=2}^{x=5} \frac{1}{\ln(x)} \cdot \frac{1}{x}\,dx = \int_{u=\ln(2)}^{u=\ln(5)} \frac{1}{u}\,du.$$

- Antidifferentiate in u:

$$\int_{\ln(2)}^{\ln(5)} \frac{1}{u}\,du = [\ln(u)]_{\ln(5)}^{\ln(2)} = \ln(\ln(5)) - \ln(\ln(2)).$$

Remember to check by differentiating !