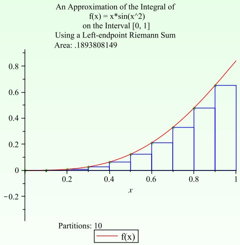
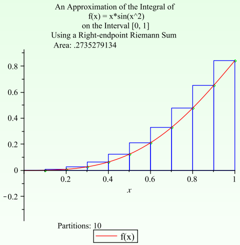
Let $I = \int_0^1 x\sin(x^2)\,dx$

1. Use the RiemannSum command in Maple to look at L_{10} and R_{10}.
   - Load the student package: *Tools-Load Package-Student Calculus 1*.
   - Type in:

```
f:= x -> x*sin(x^2);
RiemannSum(f(x), x=0..1, partition=10,method=left,
   output=plot);
RiemannSum(f(x),x=0..1, partition=10, method=right,
   output=plot);
```



L_{10}



R_{10}

2. Write L_{10} and L_{50} using sigma notation (without using Maple).

- 10 subintervals on $[0,1] \Rightarrow$ base $\Delta x = \dfrac{b-a}{n} = \dfrac{1-0}{10} = \dfrac{1}{10}$.
- Partition: $0 < \frac{1}{10} < \frac{2}{10} < \frac{3}{10} < \ldots < \frac{9}{10} < 1$.
- Left sum $\Rightarrow$ heights determined left endpoints

$$0, \frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \ldots, \frac{9}{10}.$$

- heights of the rectangles are $f\left(\frac{i}{10}\right)$ for $i = 0 \ldots 9$.
- Area of rectangle $=$ base $\times$ height, so

$$L_{10} = \frac{1}{10}\left(f(0) + f\left(\frac{1}{10}\right) + f\left(\frac{2}{10}\right) + \ldots + f\left(\frac{9}{10}\right)\right).$$

- For sigma notation, look for pattern of consecutive numbers appearing,

$$\begin{aligned} L_{10} &= \frac{1}{10}\sum_{i=0}^{9} f\left(\frac{i}{10}\right) \\ &= \frac{1}{10}\sum_{i=0}^{9} \frac{i}{10}\sin\left(\frac{i^2}{100}\right) \end{aligned}$$

2. (continued)
As for L_{50}, what changes? The width of the subintervals, and therefore the endpoints are the main changes (the heights of the rectangles change also, but that comes for free with changing the endpoints.)

$$\Delta x = \frac{1}{50} \qquad 0 < \frac{1}{50} < \frac{2}{50} < \frac{3}{50} < \ldots < \frac{49}{50} < 1.$$

Thus the heights of the rectangles are $f\left(\frac{i}{50}\right)$ for $i = 0 \ldots 49$, and

$$\begin{aligned} L_{50} &= \frac{1}{50}\left(f(0) + f\left(\frac{1}{50}\right) + f\left(\frac{2}{50}\right) + \ldots + f\left(\frac{49}{50}\right)\right) \\ &= \frac{1}{50}\sum_{i=0}^{49} f\left(\frac{i}{50}\right) \\ &= \frac{1}{50}\sum_{i=0}^{49} \frac{i}{50}\sin\left(\frac{i^2}{2500}\right) \end{aligned}$$

3. Write R_{10} and R_{50} using Sigma notation (again without Maple).

- Partition: $0 < \frac{1}{10} < \frac{2}{10} < \frac{3}{10} < \ldots < \frac{9}{10} < 1$.
- Only difference between left and right sums: which points in the partition we use: for a right sum, use the last 10 of these same 11 points: $\frac{i}{10}$ where $i = 1..10$
- Bases: same as with L_{10}.
- Heights: still $f(\frac{i}{10})$, but $i = 1..10$.

$$R_{10} = \frac{1}{10} \sum_{i=1}^{10} \frac{i}{10} \sin\left(\frac{i^2}{100}\right)$$

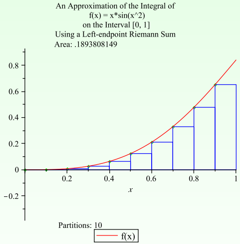
Similarly, $R_{50} = \dfrac{1}{50} \displaystyle\sum_{i=1}^{50} \dfrac{i}{50} \sin\left(\dfrac{i^2}{2500}\right)$.

3. (continued)
Compare:

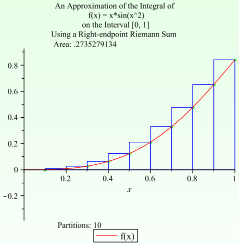
$$L_{10} = \frac{1}{10} \sum_{i=0}^{9} \frac{i}{10} \sin\left(\frac{i^2}{100}\right) \qquad R_{10} = \frac{1}{10} \sum_{i=1}^{10} \frac{i}{10} \sin\left(\frac{i^2}{100}\right)$$

All that changes is which values i ranges over!

On the graphs of L_{10} and R_{10}, this coincides to the fact that the last 9 rectangles in L_{10} are the same as the first 9 rectangles in R_{10}.



L_{10}



R_{10}

4. Use the formal definition of the integral to write $I = \int_0^1 x \sin(x^2)\, dx$ as a limit.

Using the right sum (arbitrarily),

$$\int_0^1 x \sin(x^2)\, dx \stackrel{def}{=} \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \frac{i}{n} \sin\left(\frac{i^2}{n^2}\right).$$

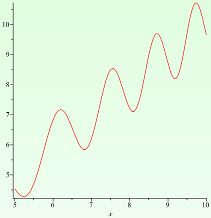
Let $I = \int_5^{10} \cos\left(\frac{x^2}{3}\right) + x\, dx$

1. Use Maple to calculate T_{1000}
   [Find left and right sums by changing "output=plot" to "output=sum"; average and approximate.]

$$
\begin{aligned}
L_{1000} &\approx 37.35741781 \\
R_{1000} &\approx 37.3802527 \\
\Rightarrow T_{1000} &= \frac{L_{1000} + R_{1000}}{2} \\
&\approx 37.37022154
\end{aligned}
$$

$$I = \int_5^{10} \cos\left(\frac{x^2}{3}\right) + x \, dx$$

2. How close is T_{1000} to the actual value of I?

- Don't know exactly how close!
- 

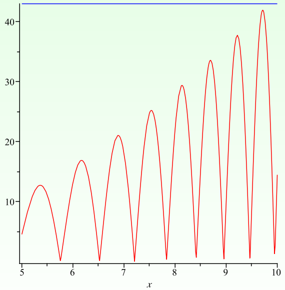
  $\cos\left(\frac{x^2}{3}\right) + x$ not always of the same concavity over $[5, 10] \Rightarrow$ can not use $|I - T_{1000}| \leq |T_{1000} - M_{1000}|$.
- $f''(x)$ continuous $\Rightarrow$ Theorem 7.1 applies.

2. How close is T_{1000} to the actual value of $I = \int_5^{10} \cos\left(\frac{x^2}{3}\right) + x\,dx$?

From Theorem 7.1, know how to bound error from using T_{1000} to approximate I:

$$\begin{aligned} \text{actual error} &\leq \text{error bound} \\ |I - T_{1000}| &\leq \frac{K(b-a)^3}{12 \cdot 1000^2} = \frac{K(10-5)^3}{1.2 \times 10^7}. \end{aligned}$$

To find K:



Graph $|f''(x)|$ over $[5, 10]$. Choose an upper bound for K a number clearly larger than (but still close to) $|f''(x)|$ over $[5, 10]$.

43 works well for K.

2. How close is T_{1000} to the actual value of $I = \int_5^{10} \cos\left(\frac{x^2}{3}\right) + x \, dx$?

$$\begin{aligned} |I - T_{1000}| &\leq \frac{43(10-5)^3}{1.2 \times 10^7} \\ &\leq .000447916667 \\ &\leq .00045 \end{aligned}$$

So we can use 37.37022154 to approximate I, and know that the **error in this approximation** is less than .00045.

$$I = \int_5^{10} \cos\left(\frac{x^2}{3}\right) + x \, dx$$

3. Determine how many subintervals n you need in order for M_n to approximate I within.0001. Find M_n using Maple.

   We *want* $|I - M_n| \leq .0001$.
   We *know* $|I - M_n| \leq \dfrac{K(b-a)^3}{24n^2}$.

   Need to find n so that $\dfrac{K(b-a)^3}{24n^2} \leq .0001$.

3. Find n so M_n approximates $II = \int_5^{10} \cos\left(\frac{x^2}{3}\right) + x\, dx$ within.0001. Find M_n.

Use the same K as I found in Problem 2.

$$\begin{aligned}
\frac{K(b-a)^3}{24n^2} &\leq .0001 \\
\frac{43 \cdot (10-5)^3}{24n^2} &\leq \frac{1}{10000} \\
\frac{5375 \cdot 10000}{24} &\leq n^2 \\
2239583.333 &\leq n^2 \\
1496.52 &\leq n
\end{aligned}$$

Therefore M_{1497} is guaranteed to be within 0.0001 of the actual value of I.