Recall: Definition and Notation for Signed Volume over a Non-Rectangular Region

- **Definition:** Let R be any region in the xy-plane bounded by a simple closed curve, and let $f(x, y)$ be a function defined on R. Then we can define the signed volume between the surface $z = f(x, y)$ and the xy-plane over the region R:

$$\text{Signed Volume} = \lim_{n\to\infty} \sum_{i=1}^{n} f(u_i, v_i)\Delta A,$$

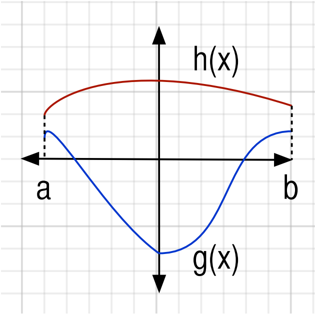
  where for $i = 1..n$, (u_i, v_i) is a point in the ith rectangle of the partition of R and ΔA is its area.

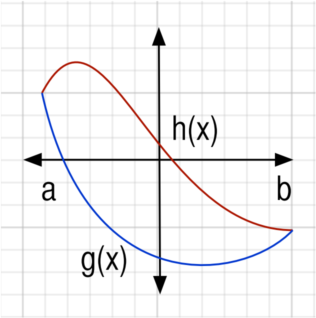
- **Notation:** We represent this signed volume by $\displaystyle\iint_R f(x, y)\, dA$, just as we did for signed volume over a rectangular region.

Recall: Fubini's Theorem for General Regions

Case 1: If R is bounded by $x = a$, $x = b$, $y = g(x)$ and $y = h(x)$ with $a \leq x \leq b$, and $g(x) \leq y \leq h(x)$ on $[a, b]$, then

$$\iint_R f(x,y)\, dA = \int_a^b \int_{g(x)}^{h(x)} f(x,y)\, dy\, dx.$$

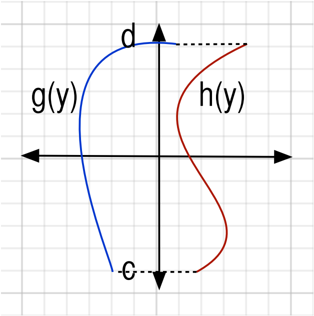


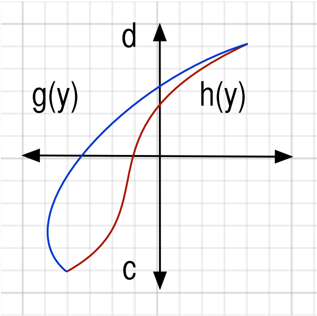


Recall: Fubini's Theorem for General Regions

Case 2: Similarly, if R is bounded by $y = a$, $y = b$, $x = g(y)$ and $x = h(y)$ with $a \leq y \leq b$, and $g(y) \leq x \leq h(y)$ on $[a, b]$, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g(y)}^{h(y)} f(x, y)\, dx\, dy.$$





Daily WW

Find the limits of integration for the following iterated integral:

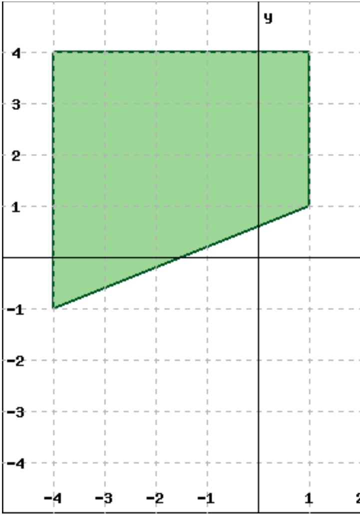
$$\iint_R f(x,y)\,dA = \int_A^B \int_C^D f(x,y)\,dy\,dx.$$

Outer integral is wrt x, so look at absolute farthest left and farthest right values.

$$-4 \le x \le 1 \Rightarrow \int_{-4}^{1} \int_C^D f(x,y)\,dy\,dx$$

For every x in $[-4,1]$, y goes from slanted line to $y=4$, so $D=4$.

For C, need to find equation of slanted line.



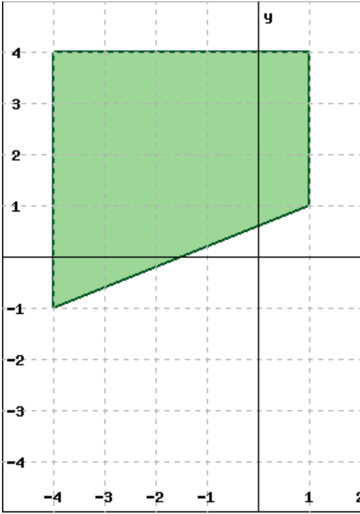
Daily WW

So far, have found:

$$\iint_R f(x,y)\,dA = \int_{-4}^{1}\int_{C}^{D} f(x,y)\,dy\,dx$$

Slanted line:

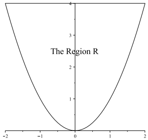
$$y+1 = \frac{1-(-1)}{1-(-4)}(x+4) \Rightarrow y = \frac{2}{5}(x+4)-1$$

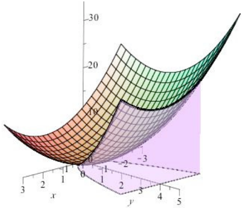


$$\iint_R f(x,y)\,dA = \int_{-4}^{1}\int_{\frac{2}{5}(x+4)-1}^{4} f(x,y)\,dy\,dx$$

Recall: Example from last time:

Let R be the region in the xy-plane bounded above by $y = 4$ and below by $y = x^2$. Find $\iint_R x^2 + y^2 \, dA$.





Two ways to think of R:

- For all $-2 \leq x \leq 2$, $x^2 \leq y \leq 4 \Rightarrow$
  $$\iint_R x^2 + y^2 \, dA = \int_{-2}^{2} \left(\int_{x^2}^{4} x^2 + y^2 \, dy \right) dx$$
- For all $0 \leq y \leq 4$, $-\sqrt{y} \leq x \leq \sqrt{y} \Rightarrow$
  $$\iint_R x^2 + y^2 \, dA = \int_{0}^{4} \left(\int_{-\sqrt{y}}^{\sqrt{y}} x^2 + y^2 \, dx \right) dy$$

In Class Work

1. Suppose that R is the region bounded by $y = 1 + x$, $y = 1 - x$, and $y = -1$, and that $f(x, y)$ is a continuous function over R. Rewrite $\displaystyle\iint_R f(x, y)\, dA$ as one or more iterated integral of the form:
   (a) $\int_A^B \int_C^D f(x, y)\, dy\, dx$
   (b) $\int_A^B \int_C^D f(x, y)\, dx\, dy$

2. Find the signed volume btwn the surface $z = 1 + x + y$ and the region R in the xy-plane bdd by the graphs $x = 1, y = 0, y = x^2$.

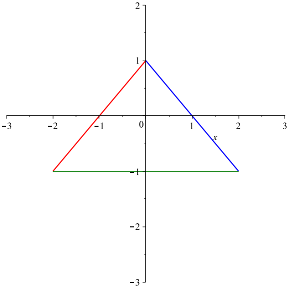
3. Find the signed volume btwn the surface $z = e^{-x^2}$ and the triangle R in the xy-plane bdd by the x-axis, the line $x = 1$, and the line $y = x$.

4. (a) Try to evaluate $\displaystyle\int_0^\pi \int_x^\pi \frac{\sin(y)}{y}\, dy\, dx$ as it's written. What happens?
   (b) Sketch the region we're integrating over.
   (c) Use the sketch from (b) to reverse the order of integration. (Remember to switch from $x = g(y)$ and $x = h(y)$ to $y = g(x)$ and $y = h(x)$.) Now try to evaluate the integral. Is this order more effective than the first?

Solutions

1. Suppose that R is the region bounded by $y = 1 + x$, $y = 1 - x$, and $y = -1$, and that $f(x, y)$ is a continuous function over R. Rewrite $\iint_R f(x, y)\, dA$ as the iterated integral:

(a) $\int_A^B \int_C^D f(x, y)\, dy\, dx$ (b) $\int_A^B \int_C^D f(x, y)\, dx\, dy$



(a) x appears at first to go from -2 to 2, but when we look at the upper and lower curves, we see that the upper curve is not always the same.

When $x \in [-2, 0]$, the lower curve is $y = -1$ and the upper curve is $y = 1 + x$. But when $x \in [0, 2]$, while the lower curve is still $y = -1$, the upper curve is now $y = 1 - x$.

Thus

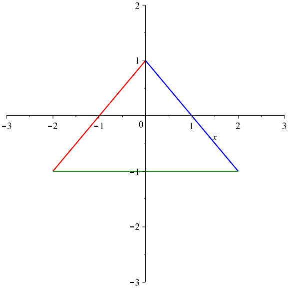
$$\iint_R f(x, y)\, dA = \int_{x=-2}^{x=0} \int_{y=-1}^{y=1+x} f(x, y)\, dy\, dx + \int_{x=0}^{x=2} \Big(\int_{y=-1}^{y=1-x} f(x, y)\, dy\, dx$$

Solutions

1. Suppose that R is the region bounded by $y = 1 + x$, $y = 1 - x$, and $y = -1$, and that $f(x, y)$ is a continuous function over R. Rewrite $\iint_R f(x, y)\, dA$ as the iterated integral:

(a) $\int_A^B \int_C^D f(x, y)\, dy\, dx$

(b) $\int_A^B \int_C^D f(x, y)\, dx\, dy$



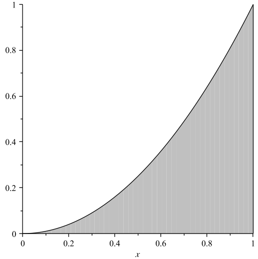
(b) y goes from -1 to 1; over that interval, x goes from left curve $x = y - 1$ to right curve $x = 1 - y$.

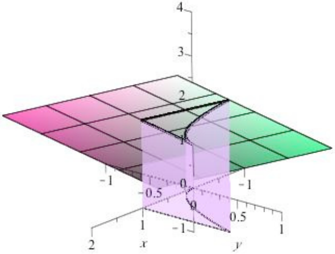
Thus

$$\iint_R f(x, y)\, dA = \int_{y=-1}^{y=1} \int_{x=y-1}^{x=1-y} f(x, y)\, dx\, dy$$

Solutions

2. Find the signed volume between the surface $z = 1 + x + y$ and the region R in the xy-plane bounded by the graphs $x = 1, y = 0, y = x^2$.





Using sketch of R, shown on the left, notice that we can either say that

$$0 \le x \le 1 \text{ and } 0 \le y \le x^2$$

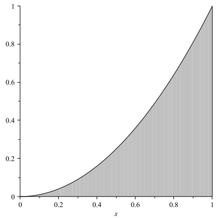
or

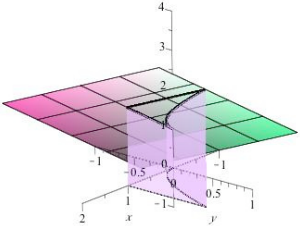
$$0 \le y \le 1 \text{ and } \sqrt{y} \le x \le 1.$$

$$\boxed{\text{I will choose to use } 0 \le x \le 1 \text{ and } 0 \le y \le x^2}$$

Solutions

2. Find the signed volume between the surface $z = 1 + x + y$ and the region R in the xy-plane bounded by the graphs $x = 1, y = 0, y = x^2$.





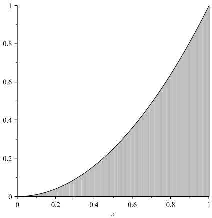
Choosing to use $0 \le x \le 1$ and $0 \le y \le x^2$

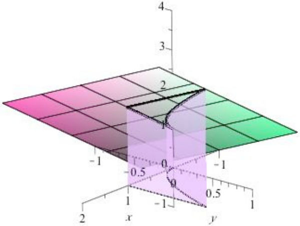
Thus by Fubini's Theorem,

$$V = \iint_R 1 + x + y \, dA = \int_0^1 \left(\int_0^{x^2} 1 + x + y \, dy \right) \, dx.$$

Solutions

2. Find the signed volume between the surface $z = 1 + x + y$ and the region R in the xy-plane bounded by the graphs $x = 1, y = 0, y = x^2$.



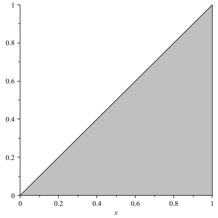


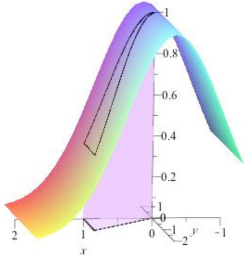
Thus by Fubini's Theorem,

$$
\begin{aligned}
V &= \iint_R 1 + x + y \, dA = \int_0^1 \left(\int_0^{x^2} 1 + x + y \, dy \right) dx \\
&= \int_0^1 \left(y + xy + \frac{y^2}{2} \Big|_0^{x^2} \right) dx \\
&= \int_0^1 x^2 + x^3 + \frac{1}{2} x^4 \, dx = \ldots = \frac{41}{60}
\end{aligned}
$$

Solutions

3. Find the volume below the surface $z = e^{-x^2}$ and above the triangle R in the xy-plane bounded by the x-axis, the line $x = 1$, and the line $y = x$.





Again, using sketch of region R on left, write the region in two ways :

$$0 \le x \le 1 \text{ and } 0 \le y \le x$$

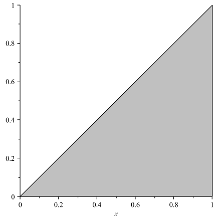
or

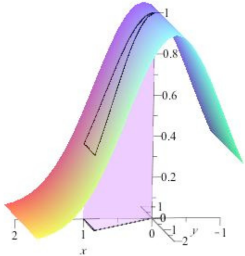
$$0 \le y \le 1 \text{ and } y \le x \le 1.$$

$$\boxed{\text{I will choose to use } 0 \le x \le 1 \text{ and } 0 \le y \le x}$$

Solutions

3. Find the volume below the surface $z = e^{-x^2}$ and above the triangle R in the xy-plane bounded by the x-axis, the line $x = 1$, and the line $y = x$.





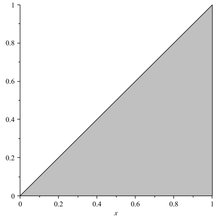
Choosing to use $0 \leq x \leq 1$ and $0 \leq y \leq x$

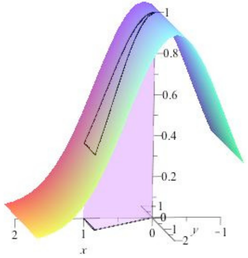
Thus by Fubini's Theorem,

$$V = \iint_R e^{-x^2}\, dA = \int_0^1 \left(\int_0^x e^{-x^2}\, dy \right)\, dx$$

Solutions

3. Find the volume below the surface $z = e^{-x^2}$ and above the triangle R in the xy-plane bounded by the x-axis, the line $x = 1$, and the line $y = x$.





Thus by Fubini's Theorem,

$$\begin{aligned} V &= \iint_R e^{-x^2}\, dA = \int_0^1 \left(\int_0^x e^{-x^2}\, dy \right) dx \\ &= \int_0^1 \left(y e^{-x^2} \Big|_0^x \right) dx = \int_0^1 x e^{-x^2}\, dx \\ &\overset{u-sub}{=} \ldots = -\frac{1}{2}\left(\frac{1}{e} - 1\right) = -\frac{1}{2e} + \frac{1}{2} \end{aligned}$$

Solutions

4.(a)

Try to evaluate $\int_0^\pi \int_x^\pi \frac{\sin(y)}{y}\, dy\, dx$ as it's written. What happens?

$$\int_0^\pi \int_x^\pi \frac{\sin(y)}{y}\, dy\, dx = \int_0^\pi \int_x^\pi \frac{1}{y} \sin(y)\, dy\, dx$$

I can already see that neither substitution nor integration by parts is going to work wonders on this integral.

When I try to do the inner integral on Maple, I get some function called "Si(y)", which I've never heard of ... or at least, it rings no bells.

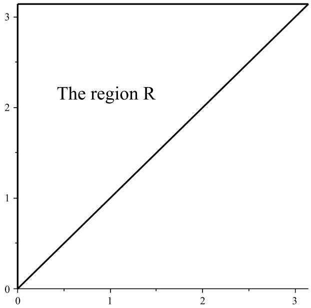
Solutions

4(b) Sketch the region we're integrating over.

In order to sketch the region, I need to look at what intervals I'm integrating over.

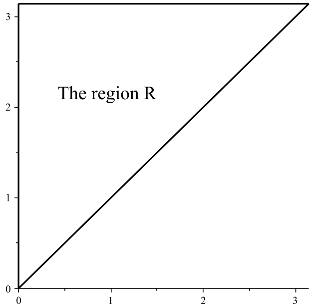
Looking at the integral $\int_0^{\pi} \left(\int_x^{\pi} \frac{\sin(y)}{y} \, dy \right) \, dx$, we see that $0 \le x \le \pi$ and $x \le y \le \pi$.

In other words, as x goes from 0 to π, y goes from the diagonal lines $y = x$ up to the horizontal line $y = \pi$.



Solutions

4(c) Reverse the order of integration (using the sketch you developed in (b)), and try to evaluate the integral. Is this way more effective than the first?



Looking at the region R, I can see that we can also say that

$$0 \leq y \leq \pi \text{ and } 0 \leq x \leq y.$$

We can thus rewrite the integral:

$$\int_0^\pi \left(\int_0^y \frac{\sin(y)}{y} \, dx \right) \, dy.$$

Because this time we're integrating with respect to x first, and because our integrand is constant with respect to x, this is suddenly much easier!

Solutions

4(c) Reverse the order of integration (using the sketch you developed in (b)), and try to evaluate the integral. Is this way more effective than the first?

We have just found that

$$\int_0^\pi \int_0^y \frac{\sin(y)}{y}\, dy\, dx = \int_0^\pi \left(\int_0^y \frac{\sin(y)}{y}\, dx \right) dy.$$

The integral on the left is a very difficult integral for us. As for the integral on the *right* ...

$$\begin{aligned}
\int_0^\pi \left(\int_0^y \frac{\sin(y)}{y}\, dx \right) dy &= \int_0^\pi \left(\frac{\sin(y)}{y} \int_0^y 1\, dx \right) dy \\
&= \int_0^\pi \left(\frac{\sin(y)}{y} (x) \Big|_0^y \right) dy \\
&= \int_0^\pi \frac{\sin(y)}{y} (y-0)\, dy = \int_0^\pi \sin(y)\, dy \\
&= -\cos(y)\Big|_0^\pi = -(-1) - (-1) = 2
\end{aligned}$$