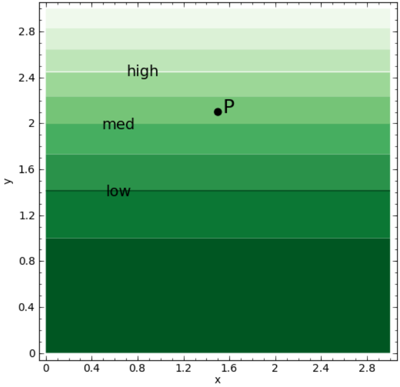
Daily WW, Problem 2



Use the level curves of the function $z = f(x, y)$ to determine if each partial derivative at the point P is positive, negative, or zero.

1. $f_x(P)$: level curves horizontal $\Rightarrow f$ constant in the x direction $\Rightarrow$ $f_x(P) = 0$

2. $f_y(P)$: as we move in the y direction, level curves - each of which represent a value of z - go from low to high. Thus z is increasing, so $f_y(P) > 0$.

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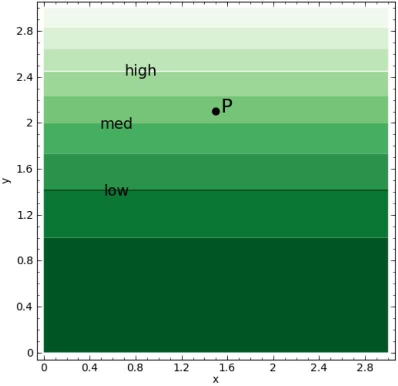


Use the level curves of the function $z = f(x, y)$ to determine if each partial derivative at the point P is positive, negative, or zero.

3. $f_{xx}(P)$: Since $f_x = 0$ not just at P but for the entire horizontal line through P, $f_{xx} = 0$.

4. $f_{xy}(P)$: Similarly, $f_x = 0$ not just at P but for the entire vertical line through P, so $f_{xy} = 0$ as well

5. $f_{yy}(P)$: Because the level curves are getting closer together as we move in the y direction, f is increasing faster and faster, and so $f_{yy} > 0$

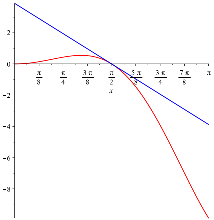
Recall:

If $f(x)$ is a differentiable function of one variable, we can use the tangent line at $x = x_0$,

$$y = L(x) = f'(x_0)(x - x_0) + f(x_0)$$

as a **linear approximation** of f at x_0.

We can use $L(x)$ to approximate $f(x)$ at points near x_0.

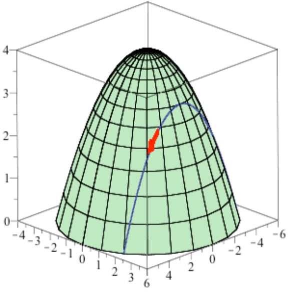


Example: When $f(x) = x^2 \cos(x)$, $f'(x) = 2x\cos(x) - x^2 \sin(x)$, so the tangent line at $x_0 = \dfrac{\pi}{2}$ has equation $L(x) = -\frac{\pi^2}{4}(x - \dfrac{\pi}{2})$.

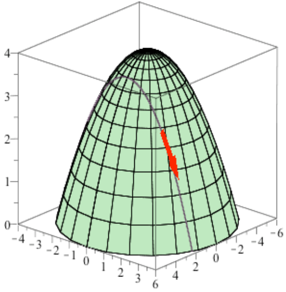
Thus

$$\begin{aligned} f(1.5) &\approx L(1.5) \\ &\approx -\frac{\pi^2}{4}(1.5 - \frac{\pi}{2}) \approx 0.175 \end{aligned}$$

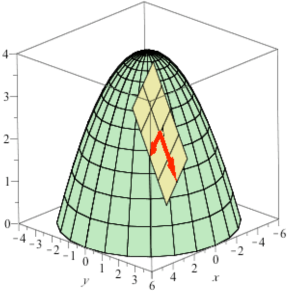
Tangent Planes



Blue curve = intersection of plane $y = b$ with surface $z = f(x, y) = z = f(x, b)$. Red arrow represents the line tangent to this curve. It has slope $f_x(a, b)$.



Brown curve = intersection of plane $x = a$ with surface $z = f(x, y) = z = f(a, y)$. Red arrow represents line tangent to this curve. It has slope $f_y(a, b)$.



These two lines (and every other line tangent to a curve on the surface that goes through the point $(a, b, f(a, b))$ will lie on our **tangent plane**, shown in yellow.

Generalizing to 3-Space

In 2-space,

$$Ax + By = C$$

is a line

What does the generalization

$$Ax + By + Cz = D$$

represent in 3-space?

How do we represent lines in 3-space?

Is

$$Ax + By + Cz = D$$

a line in 3-space?

In Class Work

Let $f(x,y) = e^{-x^2-y^2}$.

(a) Verify that at the point $\left(0,-1,\frac{1}{e}\right)$, a tangent plane exists

(b) Find the equation of that tangent plane.

(c) Use the tangent plane found in part (b) to approximate $f(0.1,-0.9)$.

Solutions

Let $f(x,y) = e^{-x^2-y^2}$.

(a) Verify that at the point $\left(0,-1,\frac{1}{e}\right)$, a tangent plane exists

- $f_x(x,y) = -2xe^{-x^2-y^2}$ and $f_y(x,y) = -2ye^{-x^2-y^2}$
- Both are continuous everywhere, so tangent plane at $(0,-1)$ exists

(b) Find the equation of that tangent plane.

- Tangent plane: $z - f(0,-1) = f_x(0,-1)(x-0) + f_y(0-,1)(y+1)$
- $f_x(0,-1) = -2(0)e^{-0^2-(-1)^2} = 0$, $f_y(0,-1) = -2(-1)e^{-0^2-(-1)^2} = \frac{2}{e}$
- Thus at $(0,-1,\frac{1}{e})$, the tangent plane is given by the equation

$$z - \frac{1}{e} = 0(x-0) + \frac{2}{e}(y+1)$$

or

$$z = \frac{2}{e}(y+1) + \frac{1}{e} = \frac{2}{e}y + \frac{3}{e}.$$

(c) Use the tangent plane found in part (b) to approximate $f(0.1,-0.9)$.

$$L(x,y) = \frac{2}{e}y + \frac{3}{e} \Rightarrow f(0.1,-0.9) \approx L(0.1,-0.9) = \frac{2}{e}(-0.9) + \frac{3}{e} = \frac{1.2}{e}$$

Solutions

Graph of $f(x, y) = e^{-x^2-y^2}$ along with tangent plane $L(x, y) = \frac{2}{e}y + \frac{3}{e}$, from two different angles.