Recall: Stationary points for a function $f(x,y)$:

If $f(x,y)$ is a smooth at (a,b) – that is, if there is a plane tangent to the surface at $\big(a,b,f(a,b)\big)$– then in order for the point $\big(a,b,f(a,b)\big)$ to be a local maximum or local minimum, it **must** be true that:

- that is, the tangent plane at (a,b) is horizontal
- $f(x,y)$ is flat at $\big(a,b,f(a,b)\big)$ in all directions
- All directional derivatives at (a,b) must be 0
- In particular $f_x(a,b)=0$ and $f_y(a,b)=0$

Thus if $f_x(a,b)\neq 0$ *or* if $f_y(a,b)\neq 0$, the point $\big(a,b,f(a,b)\big)$ can not be a local maximum or a local minimum.

To be analogous to Calc 1, we call all inputs (a,b) such that $f_x(a,b)=0$ and $f_y(a,b)=0$ *stationary points*.

Note: This does not mean that every stationary point **is** a local maximum or a local minimum.

Recall from Calc 1:

For $g(x)$, if $g'(a) = 0$ (so that g is flat at $x = a$), then the 2nd Derivative Test gives

$$g''(a) > 0 \implies g \text{ is concave up at } x = a \implies x = a \text{ is a minimum}$$
$$g''(a) < 0 \implies g \text{ is concave down at } x = a \implies x = a \text{ is a maximum}$$
$$g''(a) = 0 \implies \text{the test is inconclusive}$$

Is there an analogous result for $f(x, y)$?

Question:

Suppose $f(x, y)$ is a smooth function and

- $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$
- $f_{xx}(x_0, y_0) > 0$, $f_{yy}(x_0, y_0) > 0$ – That is, in both the x and y directions, f is concave up

Must f have a local min at the point (x_0, y_0)?

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Must f have a local min at the point (x_0, y_0)?

No!

Example:

Let $f(x,y) = x^2 + 3xy + 2y^2 - 9x - 11y$.

$$f_x(x,y) = 2x + 3y - 9 \implies f_x(-3,5) = -6 + 15 - 9 = 0$$
$$f_y(x,y) = 3x + 4y - 11 \implies f_y(-3,5) = -9 + 20 - 11 = 0$$

$$f_{xx}(x,y) = 2 \implies f_{xx}(-3,5) > 0 \qquad f_{yy}(x,y) = 4 \implies f_{yy}(-3,5) > 0$$

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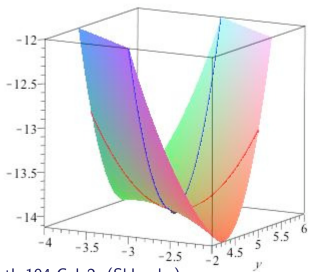
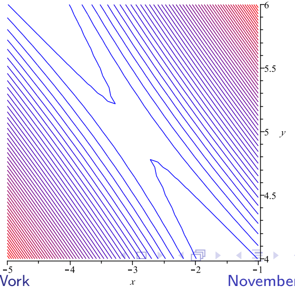
Thus at $(-3,5)$, f is flat in both the x and y directions, *and* is concave up in both the x and y directions ... *but* f does **not** have a minimum there:





Reminder: Vocabulary from Calc 1

- An *extremum* is a point on a surface which is either a maximum or a minimum. (The plural is *extrema*.)

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- f has a *local* maximum at (a, b) if $f(a, b)$ is at least as large as $f(x, y)$ for all (x, y) near (a, b). (That is, for all (x, y) in some open neighborhood containing (a, b)).

  Similarly f has a *local* minimum at (a, b) if $f(a, b)$ is at least as small as $f(x, y)$ for all (x, y) near (a, b). That is, for all (x, y) in some open neighborhood containing (a, b)).

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- f has an *absolute* maximum at (a, b) if $f(a, b)$ is at least as large as $f(x, y)$ for all (x, y) in the domain of f.

  Similarly, f has an *absolute* minimum at (a, b) if $f(a, b)$ is at least as small as $f(x, y)$ for all (x, y) in the domain of f.

Recall:

As long as $f_{xy}(x,y)$ and $f_{yx}(x,y)$ are continuous, they are equal:

$$f_{xy}(x,y) = f_{yx}(x,y)$$

In Class Work

1. Locate all critical points and classify them using the 2nd Derivatives Test:

   (a) $f(x, y) = 4xy - x^3 - 2y^2$

   (b) $f(x, y) = x^3 - 3xy + y^3$

Solutions

1. Let $f(x,y) = 4xy - x^3 - 2y^2$
(a) Find and classify all critical points of f.

$$\frac{\partial f}{\partial x} = 0 \implies 4y - 3x^2 = 0 \implies y = \frac{3}{4}x^2$$
$$\frac{\partial f}{\partial y} = 0 \implies 4x - 4y = 0 \implies y = x$$

$$x = y = \frac{3}{4}x^2 \implies x\left(1 - \frac{3}{4}x\right) = 0 \implies x = 0 \text{ or } x = \frac{4}{3}.$$

$$x = 0 \implies y = 0 \qquad x = \frac{4}{3} \implies y = \frac{4}{3}.$$

Thus our critical points are $(0,0)$ and $(4/3, 4/3)$.

Solutions

1. Let $f(x, y) = 4xy - x^3 - 2y^2$

(a) Find and classify all critical points of f.

- The critical points are $(0, 0)$ and $(4/3, 4/3)$.
- To classify them using the 2nd Derivatives Test, we need:

$$
\begin{aligned}
f_{xx}(x, y) &= \frac{\partial}{\partial x}(4y - 3x^2) = -6x \\
D(x, y) &= \det \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix} = \det \begin{bmatrix} -6x & 4 \\ 4 & -4 \end{bmatrix} = 24x - 16
\end{aligned}
$$

Solutions

1. Let $f(x, y) = 4xy - x^3 - 2y^2$

(a) Find and classify all critical points of f.

- The critical points are $(0, 0)$ and $(4/3, 4/3)$.
- $f_{xx}(x, y) = -6x$ and $D(x, y) = 24x - 16$
- At $(0, 0)$,

  $$f_{xx}(0,0) = (-6)(0) \qquad \text{and} \qquad D(0,0) = 24(0) - 16 < 0.$$

  By the 2nd Derivatives Test, because $D < 0$, f has a saddle point at $(0, 0)$.

Solutions

1. Let $f(x,y) = 4xy - x^3 - 2y^2$
(a) Find and classify all critical points of f.

- The critical points are $(0,0)$ and $(4/3, 4/3)$.
- $f_{xx}(x,y) = -6x$ and $D(x,y) = 24x - 16$
- f has a saddle point at $(0,0)$.
- At $(4/3, 4/3)$,

$$f_{xx}\left(\frac{4}{3},\frac{4}{3}\right) = (-6)\left(\frac{4}{3}\right) < 0 \quad \text{and} \quad D\left(\frac{4}{3},\frac{4}{3}\right) = 24\left(\frac{4}{3}\right) - 16 > 0.$$

By the 2nd Derivatives Test, because $D > 0$ and $f_{xx} < 0$, f has a local maximum.

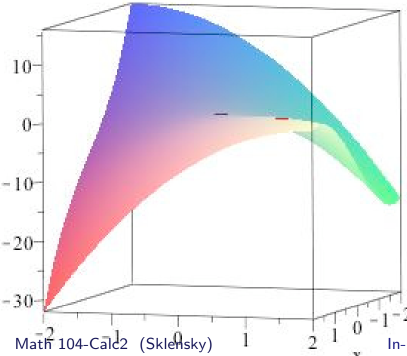
Solutions

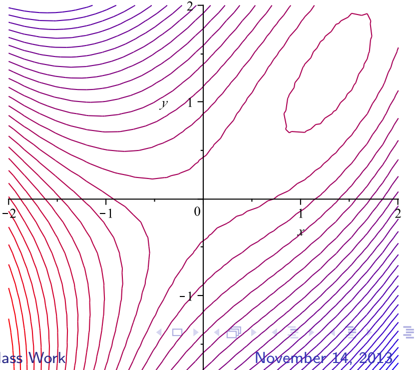
1. Let $f(x,y) = 4xy - x^3 - 2y^2$

(a) Find and classify all critical points of f.

- The critical points are $(0,0)$ and $(4/3, 4/3)$.
- $f_{xx}(x,y) = -6x$ and $D(x,y) = 24x - 16$
- f has a saddle point at $(0,0)$.
- f has local maximum at $(4/3, 4/3)$.

To illustrate these points:





Solutions

1. Locate all critical points and classify them using the 2nd Derivatives Test:
(b) $f(x,y) = x^3 - 3xy + y^3$

$$\frac{\partial f}{\partial x} = 0 \implies 3x^2 - 3y = 0 \implies y = x^2$$
$$\frac{\partial f}{\partial y} = 0 \implies -3x + 3y^2 = 0 \implies x = y^2$$

Putting those two results together,

$$y = x^2 = (y^2)^2 \implies y(y^3 - 1) = 0 \implies y = 0, y = 1$$

Thus our critical points are $(0,0)$ and $(1,1)$.

Solutions

1. Locate all critical points and classify them using the 2nd Derivatives Test:
(b) $f(x, y) = x^3 - 3xy + y^3$

- We found the only critical points are $(0, 0)$ and $(1, 1)$
- To classify them, we will use the Second Derivatives Test.

  To use that, we need to find $f_{xx}(a, b)$ and the discriminant $D(a, b)$:

$$f_{xx}(x, y) = \frac{\partial}{\partial x}(3x^2 - 3y) = 6x$$

$$D(x, y) = \det \begin{bmatrix} f_{xx}(x, y) & f_{xy}(x, y) \\ f_{xy}(x, y) & f_{yy}(x, y) \end{bmatrix} = \det \begin{bmatrix} 6x & -3 \\ -3 & 6y \end{bmatrix} = 36xy - 9$$

Solutions

1. Locate all critical points and classify them using the 2nd Derivatives Test:
(b) $f(x, y) = x^3 - 3xy + y^3$

- We found the only critical points are $(0, 0)$ and $(1, 1)$
- Also found $f_{xx}(x, y) = 6x$ and $D(x, y) = 36xy - 9$
- At $(0, 0)$,

$$f_{xx}(0, 0) = (6)(0) = 0 \quad \text{and} \quad D(0, 0) = (36)(0)(0) - 9 < 0.$$

According to the Second Derivatives Test, because $D < 0$, f has a saddle point at the point $(0, 0)$.

Solutions

1. Locate all critical points and classify them using the 2nd Derivatives Test:

(b) $f(x, y) = x^3 - 3xy + y^3$

- We found the only critical points are $(0, 0)$ and $(1, 1)$
- We found $(0, 0)$ is a saddle point.
- $f_{xx}(x, y) = 6x$ and $D(x, y) = 36xy - 9$
- At $(1, 1)$,

$$f_{xx}(1,1) = (6)(1) > 0 \quad \text{and} \quad D(1,1) = (36)(1)(1) - 9 > 0.$$

According to the Second Derivatives Test, because $D > 0$ *and* $f_{xx} > 0$, f has a local minimum at $(1, 1)$.

Solutions

1. Locate all critical points and classify them using the 2nd Derivatives Test:

(b) $f(x,y) = x^3 - 3xy + y^3$

- We found the only critical points are $(0,0)$ and $(1,1)$
- We found $(0,0)$ is a saddle point and $(1,1)$ is a local minimum.

We can see these results on a contour plot of $f(x,y)$: