# Daily WW Problem 2

Suppose $z = x^2 \sin(y)$, $x = 2s^2 + 2t^2$, $y = 2st$.

A. Use the chain rule to find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ as functions of $x$, $y$, $s$ and $t$.

$$\begin{aligned}\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \\ &= 2x\sin(y)4s + x^2\cos(y)2t\end{aligned}$$

$$\begin{aligned}\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} \\ &= 2x\sin(y)4t + x^2\cos(y)2s\end{aligned}$$

# Daily WW Problem 2 (continued)

Suppose $z = x^2 \sin(y)$, $x = 2s^2 + 2t^2$, $y = 2st$.

B. Find the numerical values of $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ when $(s, t) = (-3, 3)$.

$$\begin{aligned}
\frac{\partial z}{\partial s} &= 2x\sin(y)4s + x^2\cos(y)2t \\
&= \left[2(2\cdot 9 + 2\cdot 9)\sin(2\cdot -3\cdot 3)\right](4\cdot -3) \\
&\qquad + \left(2\cdot 9 + 2\cdot 9\right)^2\cos(2\cdot -3\cdot 3))\left(2\cdot 3\right)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial z}{\partial t} &= 2x\sin(y)4t + x^2\cos(y)2s \\
&= \left[2(2\cdot 9 + 2\cdot 9)\sin(2\cdot -3\cdot 3)\right](4\cdot 3) \\
&\qquad + \left(2\cdot 9 + 2\cdot 9\right)^2\cos(2\cdot -3\cdot 3))\left(2\cdot -3\right)
\end{aligned}$$

# Where we've been, where we're headed:

- **Recall:**
  - On a smooth surface $z = f(x, y)$, extreme can **only** occur at a point $\big(a, b, f(a, b)\big)$ **if** $f_x(a, b) = 0 = f_y(a, b)$.
  - Pts $(a, b)$ with $f_x(a, b) = 0 = f_y(a, b)$ are stationary pts. Not all stationary pts **will** be local extrema. Those that aren't are saddle pts.
  - Use the $2^{nd}$ Derivatives Test to classify each stationary point as Local Minimum, Local Maximum, Saddle Point (where possible).
- **We will see:**
  - Just as with functions of a single variable, we have an Extreme Value Theorem that guarantees that, under certain conditions, on a closed and bounded region $f$ will have both an absolute maximum and an absolute minimum value.
  - Just as with functions of a single variable, the absolute or global max and absolute or global min may occur at a local max or local min, or may occur on the boundary of the closed region you're considering.

# Recall: In Class Work from Last Time

1(a). Let $f(x, y) = 4xy - x^3 - 2y^2$. Find and classify all critical points of $f$.

Last time found, using the Second Derivatives Test:

- Critical points: $(0, 0)$ and $(4/3, 4/3)$.
- $(0, 0)$ is a saddle point
- $(4/3, 4/3)$ is a local maximum

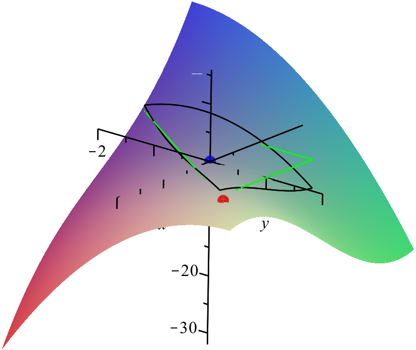
**What if instead, I ask you to find the absolute maximum value $f(x, y)$ attains on or with the triangle connecting $(1, 1)$, $(-1, 1)$, and $(-1, -2)$?**

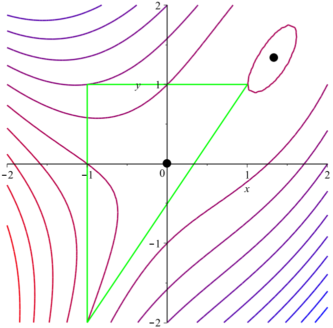
# Absolute Extrema

Let $f(x, y) = 4xy - x^3 - 2y^2$.

**Find the absolute maximum value $f(x, y)$ attains on or within the triangle connecting $(1, 1)$, $(-1, 1)$, and $(-1, -2)$.**

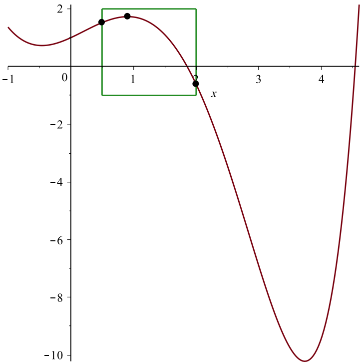
**Know:** Only Critical Points: $(0, 0)$ (saddle point); $(4/3, 4/3)$ (local max)



# With a function of a single variable:

To find the absolute extrema of a function $f(x)$ on an interval $[a, b]$:

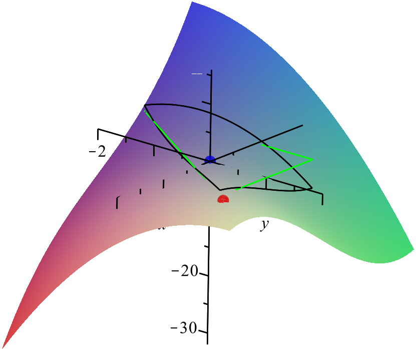


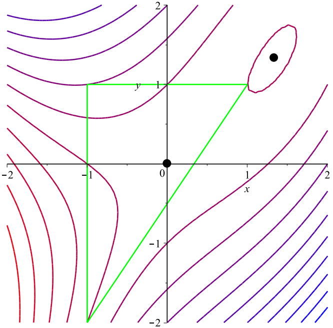

- Find critical points of $f$:
  Where is $f'(x) = 0$? Where d.n.e.?
- Relevant critical points?
  Critical points within $(a, b)$.
- Determine absolute extrema:
  Evaluate $f(x)$ at each critical point, and $f(a)$, $f(b)$.
  - Largest value of $f =$ absolute maximum value on $[a, b]$
  - Smallest value of $f =$absolute minimum value on $[a, b]$

# Absolute Extrema

Let $f(x,y) = 4xy - x^3 - 2y^2$.
Find the absolute maximum value $f(x,y)$ attains on or within the triangle connecting $(1,1)$, $(-1,1)$, and $(-1,-2)$.

**Know: Only Critical Points:** $(0,0)$ (saddle point); $(4/3, 4/3)$ (local max)



# In Class Work

Let $f(x,y) = 4xy - x^3 - 2y^2$. Find the absolute max value that $f(x,y)$ attains on or in the triangle connecting $(1,1)$, $(-1,1)$, and $(-1,-2)$.

(a) Find all critical points (stationary pts + pts where $f$ isn't diff'ble)

*Found last class: $f$ diff'ble for all $x$; only stationary points are $(0,0)$ (saddle point); $(4/3, 4/3)$ (local max)*

(b) Find a way to write boundary mathematically in a way that can be plugged into $f(x,y)$.

*The bdry consists of lines $y = 1$ for $-1 \leq x \leq 1$, $x = -1$ for $-2 \leq y \leq 1$, and $y = \frac{3}{2}x - \frac{1}{2}$*

(c) Use Calc 1 techniques to find the critical points of the function just on the boundary – that is, optimize the function you found in part (b).

(d) List all critical points within or on the triangle. Evaluate $f$ at each. Determine which gives the absolute max and which gives the absolute min.

# Solutions

Let $f(x, y) = 4xy - x^3 - 2y^2$. Find the absolute maximum value that $f(x, y)$ attains on or within the triangle connecting $(1, 1)$, $(-1, 1)$, and $(-1, -2)$.

(a) Find all critical points (stationary pts + pts where $f$ isn't diff'ble)

*Found last class: f diff'ble for all x; only stationary points are* $(0, 0)$ *(saddle point);* $(4/3, 4/3)$ *(local max)*

Since $(4/3, 4/3)$ is outside the triangle, the absolute maximum value must occur somewhere on the **boundary** of the region – that is, either on the interior of one of the edges or at one of the corners.

(b) Find a way to write boundary mathematically in a way that can be plugged into $f(x, y)$.

*Boundary:* $y = 1$ *for* $-1 \leq x \leq 1$, $x = -1$ *for* $-2 \leq y \leq 1$, *and* $y = \frac{3}{2}x - \frac{1}{2}$ *for* $-1 \leq x \leq 1$

$\Longrightarrow$ We want to maximize f(x,y) along these three curves

# Solutions

Let $f(x,y) = 4xy - x^3 - 2y^2$. Find the absolute maximum value that $f(x,y)$ attains on or within the triangle connecting $(1,1)$, $(-1,1)$, and $(-1,-2)$.

- Only critical points: $(0,0)$ (saddle point); $(4/3, 4/3)$ (local max).
- Absolute max on/within triangle must occur on the boundary.
- So want to find the maximum height that these three curves achieve:
  - $g(x) = f(x,1) = 4x - x^3 - 2$ on $-1 \leq x \leq 1$
  - $h(y) = f(-1,y) = -4y + 1 - 2y^2$ on $-2 \leq y \leq 1$
  - $j(x) = f\left(x, \dfrac{3}{2}x - \dfrac{1}{2}\right) = 4x\left(\frac{3}{2}x - \dfrac{1}{2}\right)$

(c) Use Calc 1 techniques to find the critical points of the function just on the boundary – that is, find the critical points of $g(x)$ on $[-1,1]$, $h(y)$ on $[-2,1]$, and $j(x)$ on $[-1,1]$ .

# Solutions

Let $f(x, y) = 4xy - x^3 - 2y^2$. Find the absolute maximum value that $f(x, y)$ attains on or within the triangle connecting $(1, 1)$, $(-1, 1)$, and $(-1, -2)$.

- Only critical points: $(0, 0)$ (saddle point); $(4/3, 4/3)$ (local max).
- Absolute max on/within triangle must occur on the boundary.

(c) Use Calc 1 techniques to find the critical points of the function just on the boundary – that is, find the critical points of $g(x)$ on $[-1, 1]$, $h(y)$ on $[-2, 1]$, and $j(x)$ on $[-1, 1]$ .

$$\begin{aligned} g(x) &= 4x - x^3 - 2 \text{ on } -1 \le x \le 1 \\ g'(x) &= 4 - 3x^2 \Rightarrow g'(x) = 0 \text{ when } x = \pm\frac{2}{\sqrt{3}} \end{aligned}$$

Both of these critical points lie outside of $-1 \le x \le 1$, so the only points we need to consider are the endpoints $x = -1$ and $x = 1$.

# Solutions

Let $f(x, y) = 4xy - x^3 - 2y^2$. Find the absolute maximum value that $f(x, y)$ attains on or within the triangle connecting $(1, 1)$, $(-1, 1)$, and $(-1, -2)$.

- Only critical points: $(0, 0)$ (saddle point); $(4/3, 4/3)$ (local max).
- Absolute max on/within triangle must occur on the boundary. First edge has critical points $(-1, 1)$ and $(1, 1)$

(c) Use Calc 1 techniques to find the critical points of the function just on the boundary – that is, find the critical points of $g(x)$ on $[-1, 1]$, $h(y)$ on $[-2, 1]$, and $j(x)$ on $[-1, 1]$ .

$$\begin{aligned} h(y) &= -4y + 1 - 2y^2 \text{ on } -2 \leq y \leq 1 \\ h'(y) &= -4 - 4y \Rightarrow h'(y) = 0 \text{ when } y = -1 \end{aligned}$$

$y = -1$ lies within $-2 \leq y \leq 1$, so new critical points are $y = -2$, $y = -1$, and $y = 1$.

# Solutions

Let $f(x,y) = 4xy - x^3 - 2y^2$. Find the absolute maximum value that $f(x,y)$ attains on or within the triangle connecting $(1,1)$, $(-1,1)$, and $(-1,-2)$.

- Only critical points: $(0,0)$ (saddle point); $(4/3, 4/3)$ (local max).
- Absolute max on/within triangle must occur on the boundary. First edge has critical points $(-1,1)$ and $(1,1)$; second at $(-1,-2)$, $(-1,-1)$, and $(-1,1)$ (a repeat)

(c) Use Calc 1 techniques to find the critical points of the function just on the boundary – that is, find the critical points of $g(x)$ on $[-1,1]$, $h(y)$ on $[-2,1]$, and $j(x)$ on $[-1,1]$ .

$$\begin{aligned} j(x) &= 6x^2 - 2x \text{ on } -1 \leq x \leq 1 \\ j'(x) &= 12x - 2 \Rightarrow j'(x) = 0 \text{ when } x = \frac{1}{6} \end{aligned}$$

$x = \frac{1}{6}$ lies within $-1 \leq x \leq 1$, so new critical points are $x = -1$, $x = \frac{1}{6}$, and $x = 1$.

# Solutions

Let $f(x, y) = 4xy - x^3 - 2y^2$. Find the absolute maximum value that $f(x, y)$ attains on or within the triangle connecting $(1, 1)$, $(-1, 1)$, and $(-1, -2)$.

- Only critical points: $(0, 0)$ (saddle point); $(4/3, 4/3)$ (local max).
- Absolute max on/within triangle must occur on the boundary.

(d) List all critical points within or on the boundary. Evaluate $f$ at each. Determine which gives the absolute max and which the absolute min.
The three edges have critical points $(-1, 1)$, $(1, 1)$, $(-1, -2)$, $(-1, -1)$, $\left(\frac{1}{6}, \frac{3}{2} \cdot \frac{1}{6} - \frac{1}{2}\right)$. The interior has critical point $(0, 0)$, which we already know is a saddle point so can't be abs max or abs min.

$f(-1, 1) = -5$ $\quad f(1, 1) = 1$ $\quad f(-1, -2) = 1$

$f(-1, -1) = 3$ $\quad f\left(\frac{1}{6}, -\frac{1}{4}\right) = -\frac{8}{27}$ $\quad f(0, 0) = 0$

# Solutions

Let $f(x,y) = 4xy - x^3 - 2y^2$. Find the absolute maximum value that $f(x,y)$ attains on or within the triangle connecting $(1,1)$, $(-1,1)$, and $(-1,-2)$.

Found that maximum value of $f$ on/in the triangle occurs at approximately $(-1,-1)$. As a bonus, we found that the minimum occurs at $(-1,1)$