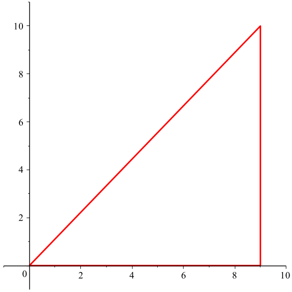
Recall: The Extreme Value Theorem (2 variables)

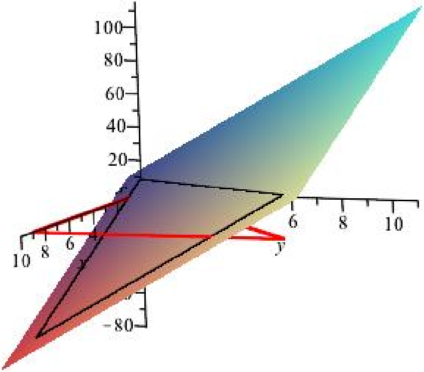
Suppose that $f(x, y)$ is **continuous** on the **closed and bounded** region $R \subset \mathbb{R}^2$. Then

- f has both an absolute maximum and an absolute minimum on R.
- an absolute extremum may only occur at a critical point (stationary point or point where f exists but isn't diff'ble) in R **or** at a point on the boundary of the region R.

Daily WW

Find the absolute minimum and absolute maximum of $f(x, y) = 8 - 8x + 9y$ on the closed triangular region with vertices $(0, 0)$, $(9, 0)$, and $(9, 10)$.





Daily WW

Find the absolute minimum and absolute maximum of $f(x,y) = 8 - 8x + 9y$ on the closed triangular region with vertices $(0,0)$, $(9,0)$, and $(9,10)$.

- Find the critical points of $f(x,y)$; keep all that lie in or on the boundary of the triangle.
  - Find the first partial derivatives
$$f_x(x,y) = -8, f_y(x,y) = 9$$
  - Because the partials are continuous everywhere, $f(x,y)$ is differentiable everywhere. Thus the only critical points are the stationary points - the points where the first partials are simultaneously 0.
  - However the first partials are never 0, and so there are no critical points from this step.

No critical points on the interior of the triangle.

Daily WW

Find the absolute minimum and absolute maximum of $f(x, y) = 8 - 8x + 9y$ on the closed triangular region with vertices $(0, 0)$, $(9, 0)$, and $(9, 10)$.

- No critical points on the interior of the triangle.
- Find the critical points on each edge of the triangle.
  - Edge 1: $y = 0$, $0 \leq x \leq 9$

$$\begin{aligned} g(x) &= f(x, 0) \\ &= 8 - 8x \\ \\ \Rightarrow g'(x) &= -8 \neq 0 \end{aligned}$$

Thus Edge 1 has no critical points on the interior of the edge. The only critical points are the boundary of the edge: the endpoints $(0, 0)$ and $(9, 0)$.

Daily WW

Find the absolute minimum and absolute maximum of $f(x, y) = 8 - 8x + 9y$ on the closed triangular region with vertices $(0, 0)$, $(9, 0)$, and $(9, 10)$.

- No critical points on the interior of the triangle.
- Find the critical points on each edge of the triangle.
  - No critical points on the interior of Edge 1: $y = 0$, $0 \leq x \leq 9$. Only the endpoints $(0, 0)$ and $(9, 0)$.
  - Edge 2: $x = 9$, $0 \leq y \leq 10$

$$
\begin{aligned}
h(y) &= f(9, y) \\
&= 8 - 72 + 9y = -64 + 9y
\end{aligned}
$$

$$
h'(y) = 9 \neq 0
$$

Thus Edge 2 has no critical points on the interior of the edge. The only critical points are the boundary of the edge: the endpoints $(9, 0)$ and $(9, 10)$.

Daily WW

Find the absolute minimum and absolute maximum of $f(x,y) = 8 - 8x + 9y$ on the closed triangular region with vertices $(0,0)$, $(9,0)$, and $(9,10)$.

- No critical points on the interior of the triangle.
- Find the critical points on each edge of the triangle.
  - No critical points on the interior of Edge 1: $y = 0$, $0 \leq x \leq 9$. Only the endpoints $(0,0)$ and $(9,0)$.
  - No critical points on the interior of Edge 2: $x = 9$, $0 \leq y \leq 10$. Only the endpoints $(9,0)$ and (9.10).
  - Edge 3: $y = \dfrac{10-0}{9-0}x$, $0 \leq x \leq 9$

$$\begin{aligned} j(x) &= f\left(x, \frac{10}{9}x\right) = 8 - 8x + 9\left(\frac{10}{9}x\right) = 8 - 18x \\ j'(x) &= -18 \neq 0 \end{aligned}$$

And again, Edge 3 has no critical points except the endpoints $(0,0)$ and $(9,10)$.

Daily WW

Find the absolute minimum and absolute maximum of $f(x, y) = 8 - 8x + 9y$ on the closed triangular region with vertices $(0, 0)$, $(9, 0)$, and $(9, 10)$.

- No critical points on the interior of the triangle.
- The only critical points on the boundary are the three vertices, $(0, 0)$, $(9, 0)$ and (9.10).
- Once we have the list of all critical points within the interior and on the boundary, evaluate $f(x, y)$ at each. The largest value **must** be the absolute maximum value, and the smallest **must** be the absolute minimum value, that f attains over the triangle.

$$f(0,0) = 8 \quad \underbrace{f(9,0) = 8 - 72 = -64}_{\text{absolute min of -64 at } (9,0)} \quad \underbrace{f(9,10) = 8 - 72 + 90 = 26}_{\text{absolute max of 26 at } (9,10)}$$

In Class Work

For each of the following surfaces $z = f(x, y)$ and regions R, find the absolute max value that $f(x, y)$ attains over the region R.

(a) $f(x, y) = 3x^2 + 24xy + 2y^3$ over the region R bounded by the cubic $y = x^3$ and the line $y = x$ in the first quadrant.

(b) $f(x, y) = 2x^2 + 4y$ over the region R consisting of the unit disk $x^2 + y^2 \leq 1$.
Hint: While you can use that R is bounded by the two curves $y = \sqrt{1 - x^2}$ and $y = -\sqrt{1 - x^2}$ for $-1 \leq x \leq 1$, you may find it easier to use that the entire boundary of the unit disk is described by $x = \cos(t)$ and $y = \sin(t)$ for $0 \leq t \leq 2\pi$.

Recall: The absolute extrema must occur at at a critical point in R or on the boundary. When looking at the boundary, remember that absolute extrema must occur at critical points on the interval in question or at an endpoint.

Solutions

1(a) Find the absolute extrema for $f(x,y) = 3x^2 + 24xy + 2y^3$ over the region R bounded by the cubic $y = x^3$ and the line $y = x$ in the first quadrant.

- Find the critical points within the region (or on the boundary)

$$\begin{aligned} f_x(x,y) = 6x + 24y \quad &\Rightarrow \quad f_x = 0 \text{ when } 6x = -24y \text{ or } x = -4y \\ f_y(x,y) = 24x + 6y^2 \quad &\Rightarrow \quad f_y = 0 \text{ when } 6y^2 = -24x \text{ or } y^2 = -4x \end{aligned}$$

$$\begin{aligned} f_x = 0 \text{ and } f_y = 0 \quad &\Rightarrow \quad y^2 = -4x = -4(-4y) \\ &\Rightarrow \quad y(y-16) = 0 \Rightarrow y = 0 \text{ or } y = 16 \end{aligned}$$

When $y = 0$, $x = 0$ as well
There is no point with $y = 16$ in our region. Thus the only critical point within our region happens to lie on the boundary: $(0,0)$.

Solutions

1(a) Find the absolute extrema for $f(x, y) = 3x^2 + 24xy + 2y^3$ over the region R bounded by the cubic $y = x^3$ and the line $y = x$ in the first quadrant.

- The only critical point "on the interior" is on the boundary: $(0, 0)$
- Find the critical points on the boundary
  $y = x^3$ and $y = x$ intersect at $x = 0$ and $x = 1$.
  - On $y = x$, $0 \leq x \leq 1$

$$\begin{aligned} g(x) = f(x, x) &= 3x^2 + 24x^2 + 2x^3 = 27x^2 + 2x^3 \\ \Rightarrow g'(x) &= 54x + 6x^2 = 6x(x + 9) \end{aligned}$$

Thus $g'(x) = 0$ when $x = 0$ or $x = -9$.
$x = -9$ is not part of the region we are considering
The only critical point is $(0, 0)$, which is one of the endpoints.
From the EVT, we know that the extrema may also occur at the other endpoint, $(1, 1)$.

Solutions

1(a) Find the absolute extrema for $f(x, y) = 3x^2 + 24xy + 2y^3$ over the region R bounded by the cubic $y = x^3$ and the line $y = x$ in the first quadrant.

- The only critical point "on the interior" is on the boundary: $(0, 0)$
- Find the critical points on the boundary
  $y = x^3$ and $y = x$ intersect at $x = 0$ and $x = 1$.
  - On $y = x$, $0 \leq x \leq 1$ the critical points are the endpts $(0, 0)$ and $(1, 1)$
  - On $y = x^3$, $0 \leq x \leq 1$

$$\begin{aligned} h(x) = f(x, x^3) &= 3x^2 + 24x^4 + 2x^9 \\ \Rightarrow h'(x) &= 6x + 96x^3 + 18x^8 = 6x(1 + 16x^2 + 3x^7) \end{aligned}$$

    Thus $h'(x) = 0$ when $x = 0$ or when $1 + 16x^2 + 3x^7 = 0$. But $1 + 16x^2 + 3x^7 \geq 1$ on $[0, 1]$ and so can not be 0 on our region of interest.
    Once again, the only critical point is $(0, 0)$, and once again, we include the other endpoint $(1, 1)$ as well.

Solutions

1(a) Find the absolute extrema for $f(x, y) = 3x^2 + 24xy + 2y^3$ over the region R bounded by the cubic $y = x^3$ and the line $y = x$ in the first quadrant.

- The only critical point "on the interior" is on the boundary: $(0, 0)$
- The only critical points on the boundary are the endpoints $(0, 0)$ and $(1, 1)$.
- Evaluate f at the critical points and determine where the absolute maximum and minimum occur.

$$f(0, 0) = 0 \qquad f(1, 1) = 29$$

Thus the absolute maximum that the function attains over the region bounded by $y = x^3$ and $y = x$ in the first quadrant is $z = 29$, at the point $(1, 1)$ and the absolute minimum is $z = 0$ at the point $(0, 0)$.

Solutions

1(b) Find the absolute extrema for $f(x, y) = 2x^2 + 4y$ over the region R consisting of the unit disk $x^2 + y^2 \leq 1$.
Hint: While you can use that R is bounded by the two curves $y = \sqrt{1 - x^2}$ and $y = -\sqrt{1 - x^2}$ for $-1 \leq x \leq 1$, you may find it easier to use that the entire boundary of the unit disk is described by $x = \cos(t)$ and $y = \sin(t)$ for $0 \leq t < 2\pi$.

- Find the critical points on the interior of R

$$\begin{aligned} f_x(x, y) = 4x \quad &\Rightarrow \quad f_x = 0 \text{ when } x = 0 \\ f_y(x, y) = 4 \quad &\Rightarrow \quad f_y \neq 0 \end{aligned}$$

Thus there are no critical points on the interior of the region.

Solutions

1(b) Find the absolute extrema for $f(x,y) = 2x^2 + 4y$ over the region R consisting of the unit disk $x^2 + y^2 \leq 1$.
Hint: While you can use that R is bounded by the two curves $y = \sqrt{1-x^2}$ and $y = -\sqrt{1-x^2}$ for $-1 \leq x \leq 1$, you may find it easier to use that the entire boundary of the unit disk is described by $x = \cos(t)$ and $y = \sin(t)$ for $0 \leq t < 2\pi$.

- There are no critical points on the interior of R
- Find the critical points on the boundary of R
  Boundary: $x(t) = \cos(t)$, $y(t) = \sin(t)$ $0 \leq t \leq 2\pi$. Thus:

$$\begin{aligned} g(t) = f(\cos(t), \sin(t)) &= 2\cos^2(t) + 4\sin(t) \\ \Rightarrow g'(t) &= -4\cos(t)\sin(t) + 4\cos(t) \\ &= 4\cos(t)\big(-\sin(t) + 1\big) \end{aligned}$$

  Thus $g'(t) = 0$ when $\cos(t) = 0$ or when $\sin(t) = 1$.
  $\cos(t) = 0$ when $t = \frac{\pi}{2}, \frac{3\pi}{2}$; $\sin(t) = 1$ when $t = \frac{\pi}{2}$
  Critical points on the boundary are $(0,1)$ and $(0,-1)$. Also $t = 0$ and $t = 2\pi$ give us $(1,0)$.

Solutions

1(b) Find the absolute extrema for $f(x, y) = 2x^2 + 4y$ over the region R consisting of the unit disk $x^2 + y^2 \leq 1$.
Hint: While you can use that R is bounded by the two curves $y = \sqrt{1 - x^2}$ and $y = -\sqrt{1 - x^2}$ for $-1 \leq x \leq 1$, you may find it easier to use that the entire boundary of the unit disk is described by $x = \cos(t)$ and $y = \sin(t)$ for $0 \leq t < 2\pi$.

- There are no critical points on the interior of R
- The critical points on the boundary are $(0, 1)$, $(0, -1)$ and $(1, 0)$.
- Determine which of the critical points found above are the absolute max and the absolute min.

$$f(0,1) = 4 \qquad f(0,-1) = -4 \qquad f(1,0) = 2$$

Thus the absolute maximum of 4 occurs at $(0, 1)$ and the absolute minimum of -4 occurs at $(0, -1)$.