Where We're Headed

- Multivariate Integration!
  - We'll begin over rectangular regions, which is the direct analog of integrating over an interval.
  - But because two dimensions adds more possibilities, Monday and the Monday after that, we'll expand to more exciting regions.

Motivating Double Integrals

Question: What does $\int_a^b f(x)dx$ measure?

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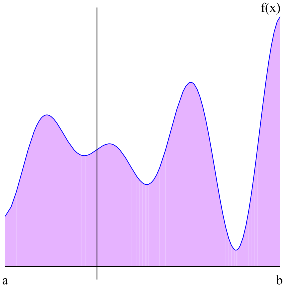

signed area between the curve and the x-axis over the interval $[a, b]$.

Motivating Double Integrals

Question: We've done multivariate differentiation. What about integration?

That is, if we think of the set of symbols $\iint \boxed{}\, dA$ as being some sort of as yet undefined multivariate integral of whatever function we put in the box, and R as a region in the xy-plane, then what might we mean by $\iint_R f(x, y)\, dA$?

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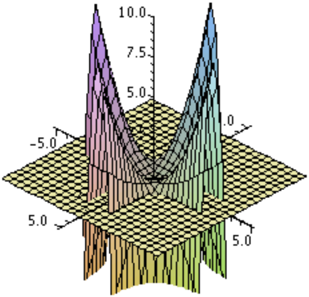


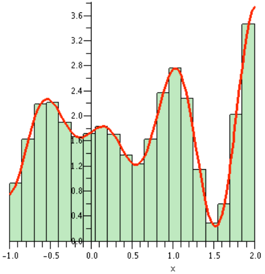
Signed volume?? between the surface and the xy-plane over the region R of the plane.

Recall from Calc 1:

To approximate the area under a positive-valued function $y = f(x)$ over the interval $I = [a, b]$, we

- **Partition the interval** $[a, b]$ into n smaller subintervals.

  Width of ith subinterval $= \Delta x_i$.



- **Pick evaluation points**, one from each subinterval. Call these c_i.

- **Add up the areas of the rectangles.**

  Base $= \Delta x_i$, height $= f(c_i) \Rightarrow$

  $$A \approx \sum_{i=1}^{n} f(c_i)\Delta x_i.$$

To find the area exactly

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x_i.$$

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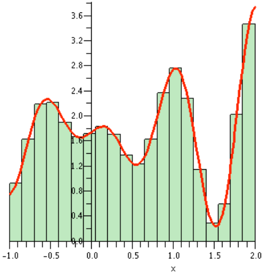

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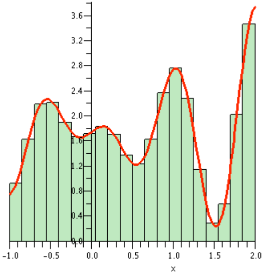

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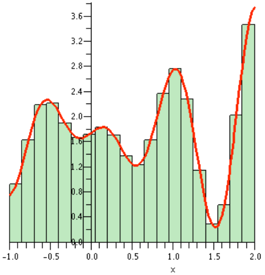

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Extend to functions of two variables, f(x, y)

We want to approximate the volume under the postiive-valued function $z = f(x, y)$ over the rectangle $R = [a, b] \times [c, d]$:

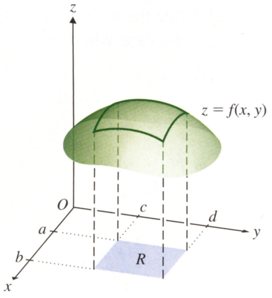

Figure 13.3
Volume under the surface $z = f(x, y)$.

Extend to $\mathbf{f(x, y)}$

- **Partition the rectangle $[\mathbf{a}, \mathbf{b}] \times [\mathbf{c}, \mathbf{d}]$ into n rectangles**, by partitioning both $[a, b]$ and $[c, d]$. Usually we'll partition both $[a, b]$ and $[c, d]$ into k subintervals, giving a total of k^2 rectangles each of area ΔA.

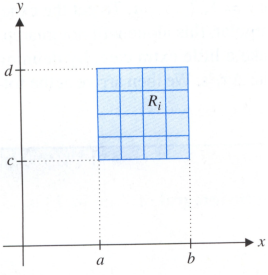

Figure 13.4a

Partition of R.

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-
-

Extend to $\mathbf{f(x, y)}$

- **Partition the rectangle $[\mathbf{a}, \mathbf{b}] \times [\mathbf{c}, \mathbf{d}]$ into n rectangles**
- **Pick evaluation points**, one from each subinterval. Call these points (u_i, v_i).
-
-

Extend to $\mathbf{f(x, y)}$

- **Partition the rectangle $[a, b] \times [c, d]$ into n rectangles**
- **Pick evaluation points**(u_i, v_i).
- **Add up the (signed) volumes of the boxes that have as their base area ΔA and as their height $f(u_i, v_9)$.**

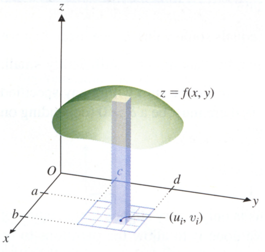


Figure 13.4b
Approximating the volume above R_i by a rectangular box.

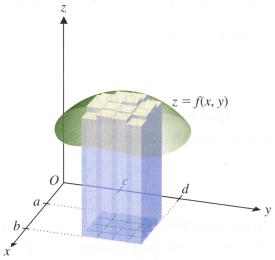


Figure 13.4c
Approximate volume.

Extend to $\mathbf{f}(\mathbf{x}, \mathbf{y})$

- **Partition the rectangle $[a, b] \times [c, d]$ into n rectangles**
- **Pick evaluation points(u_i, v_i).**
- **Add up the (signed) volumes of the boxes**
- **Take the limit as the number of boxes approaches infinity (more specifically, as the diagonal of the largest base approaches zero).**

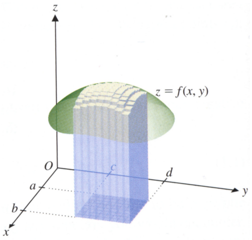
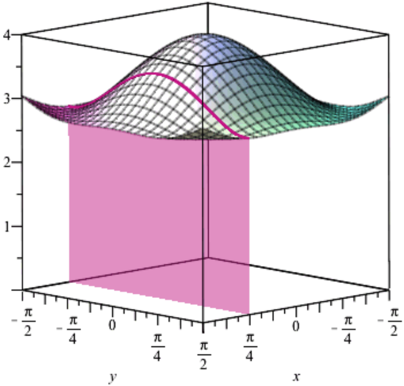


Figure 13.4d
Approximate volume.

Idea from Calc 2: Volume by Cross-Sections – Fixing x

To find $\iint_R f(x,y)\,dA =$Signed Volume over $R : [a,b] \times [c,d]$:

- For each x, take the cross-sectional signed area $A(x)$
- From Calc 2: $V = \int_a^b A(x)\,dx$
- But for any fixed x, $A(x) = \int_c^d f(x,y)\,dy$. (Keep in mind, only y acts as a variable; x is fixed)
- Thus $V = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$



Fixing y

Idea from Calc 2: Volume by Cross-Sections – Fixing x

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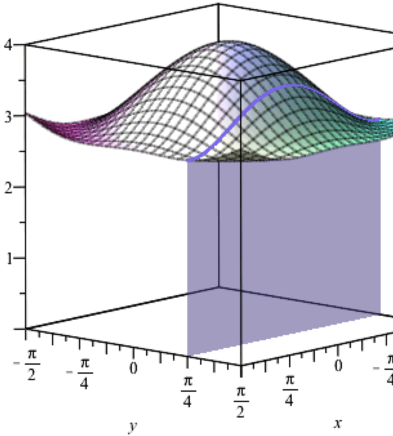

In Class Work

1. Find the volume below the surface $z = 1 + x + y$ and above the rectangle $R = \{(x, y) | 0 \leq x \leq 2, 0 \leq y \leq 3\}$ in the xy-plane.

2. Find the volume below the surface $z = y^3 e^x$ and above the rectangle $R : [-1, 1] \times [0, 2]$.

Solutions to In Class Work

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$$\begin{aligned}
\text{Signed Volume} &= \iint_R 1 + x + y \; dA \\
&= \int_0^2 \left(\int_0^3 1 + x + y \; dy \right) \; dx \\
&= \int_0^2 y + xy + \frac{1}{2} y^2 \Big|_0^3 \; dx \\
&= \int_0^2 (3 + 3x + \frac{9}{2}) - (0) \; dx = \int_0^2 \frac{15}{2} + 3x \; dx \\
&= \left(\frac{15}{2} x + \frac{3}{2} x^2 \right) \Big|_0^2 = (15 + 6) - (0 + 0) = 21
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\text{Signed Volume} &= \iint_R y^3 e^x \, dA \\
&= \int_{-1}^{1} \left(\int_0^2 y^3 e^x \, dy \right) dx \\
&= \int_{-1}^{1} \frac{y^4}{4} e^x \Big|_0^2 \, dx \\
&= \int_{-1}^{1} \frac{16}{4} e^x - 0 \, dx = \int_{-1}^{1} 4e^x \, dx \\
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