Daily WW Problem 2

2. Calculate the double integral $\iint_R x\cos(1x+y)\,dA$ where is the region: $0 \le x \le \frac{2\pi}{6}$, $0 \le y \le \frac{2\pi}{4}$.

From Fubini's Theorem, we know that

$$\begin{aligned}\iint_R x\cos(1x+y)\,dA &= \int_0^{\pi/2}\left[\int_0^{\pi/3} x\cos(1x+y)\,dx\right]dy \\ &= \int_0^{\pi/3}\left[\int_0^{\pi/2} x\cos(1x+y)\,dy\right]dx\end{aligned}$$

Sometimes, it matters which order you integrate; in this case, the order shown in the top integral is a little more efficient, but it doesn't mean the difference between being able to integrate or not being able to. Either way, we'll end up having to use integration by parts on $x\cos(x+y)$ or $x\sin(x+y)$.

If we integrate w.r.t. x first, we only have to do it on one term, while if we integrate w.r.t. y first, we'll have to do it on $x\sin(x+\pi/2)$ and on $x\sin(x)$. (I learned this by doing it both ways; if you're curious, try it!)

Daily WW Problem 2

2. (cont.) $\displaystyle\iint_R x\cos(1x+y)\,dA = \int_0^{\pi/2}\left[\int_0^{\pi/3} x\cos(1x+y)\,dx\right]dy.$

Integration by parts: Let $\begin{array}{ll} u = x & dv = \cos(x+y)\,dx \\ du = dx & v = \sin(x+y) \end{array}$

$$\begin{aligned}
\iint_R x\cos(1x+y)\,dA &= \int_0^{\pi/2}\left[\int_0^{\pi/3} x\cos(1x+y)\,dx\right]dy \\
&= \int_0^{\pi/2}\left[x\sin(x+y) - \int \sin(x+y)\,dx\right]_0^{\pi/3} dy \\
&= \int_0^{\pi/2}\left[x\sin(x+y) + \cos(x+y)\right]_0^{\pi/3} dy \\
&= \int_0^{\pi/2}\left\{\left[\frac{\pi}{3}\sin(\frac{\pi}{3}+y) + \cos(\frac{\pi}{3}+y)\right]\right. \\
&\qquad \left. - \left[0+\cos(y)\right]\right\} dy
\end{aligned}$$

Daily WW Problem 2

2. (cont.) $\iint_R x\cos(1x+y)\,dA = \int_0^{\pi/2}\left[\int_0^{\pi/3} x\cos(1x+y)\,dx\right]dy.$
Integrating with respect to x first yields:

$$\begin{aligned}
\iint_R x\cos(1x+y)\,dA &= \int_0^{\pi/2} \frac{\pi}{3}\sin(\frac{\pi}{3}+y)+\cos(\frac{\pi}{3}+y)-\cos(y)\,dy \\
&= \left[-\frac{\pi}{3}\cos(\frac{\pi}{3}+y)+\sin(\frac{\pi}{3}+y)-\sin(y)\right]_0^{\pi/2} \\
&= \left(-\frac{\pi}{3}\cos(\frac{\pi}{3}+\frac{\pi}{2})+\sin(\frac{\pi}{3}+\frac{\pi}{2})-\sin(\frac{\pi}{2})\right) \\
&\qquad -\left(\frac{\pi}{3}\cos(\frac{\pi}{3})+\sin(\frac{\pi}{3})-\sin(0)\right)
\end{aligned}$$

Recall: Signed Volume of $f(x, y)$ over $[a, b] \times [c, d]$

- Partition rectangle $[\mathbf{a}, \mathbf{b}] \times [\mathbf{c}, \mathbf{d}]$ into n sub-rectangles, by partitioning both $[a, b]$ and $[c, d]$.
- Let area of ith rectangle$=\Delta A_i$; choose point (u_i, v_i) in ith rectangle, approximate f over all of R_i by $f(u_i, v_i)$. Then signed volume $V_i = f(u_i, v_i)\Delta A_i$.
- Add up the (signed) volumes of the boxes:

$$\text{Signed V} \approx \sum_{i=1}^{n} f(u_i, v_i)\Delta A$$

- Signed V $= \lim_{n\to\infty} \sum_{i=1}^{n} f(u_i, v_i)\Delta A$

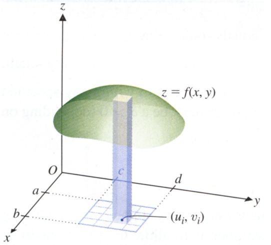
- 

Figure 13.4b
Approximating the volume above R_i by a rectangular box.

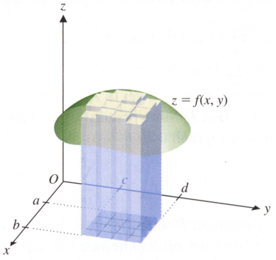
- 

Figure 13.4c
Approximate volume.

Recall: Double Integrals over Rectangle $[a, b] \times [c, d]$

- **Notation:** Let $\iint_R f(x, y)\, dA$ represent the signed volume of $f(x, y)$ over the rectangular region $R : [a, b] \times [c, d]$.

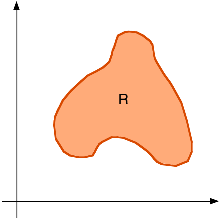
$$\iint_R f(x, y)\, dA \stackrel{def}{=} \lim_{n\to\infty} \sum_{i=1}^{n} f(u_i, v_i)\Delta A.$$

- **Fubini's Theorem**

$$\iint_R f(x, y)\, dA = \int_c^d \left(\int_a^b f(x, y)\, dx \right) dy = \int_a^b \left(\int_c^d f(x, y)\, dy \right) dx$$

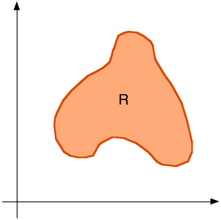
Moving beyond rectangular regions

What if our region R isn't a rectangle?



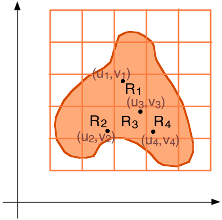
Moving beyond rectangular regions

What if our region R isn't a rectangle?



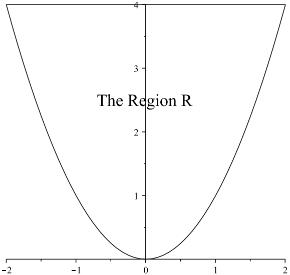
Partition the region into subrectangles.
Only consider those rectangles that lie within R.
The smaller the rectangles, the more fit entirely into the region. As we take the limit, we'll get the whole region, and the whole signed volume.

Thus we can define $\iint_R f(x,y)\ dA$ even if R isn't a rectangle.

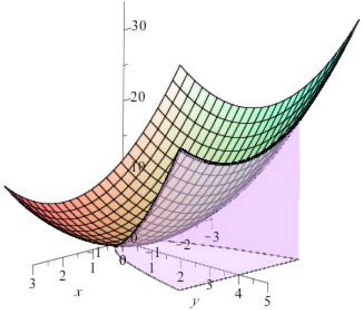


Example: Double Integral over a region which is not a Rectangle

Let R be the region in the xy-plane bounded above by $y = 4$ and below by $y = x^2$. Find $\iint_R x^2 + y^2 \, dA$.



The Region R



The surface $z = x^2 + y^2$ sitting above R and the enclosed volume

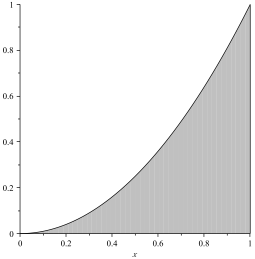
In Class Work

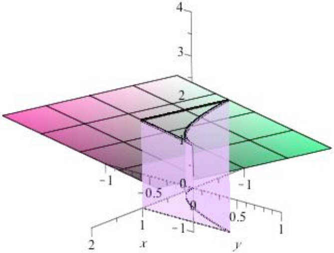
1. Find the signed volume between the surface $z = 1 + x + y$ and the region R in the xy-plane bounded by the graphs $x = 1, y = 0, y = x^2$.

2. Find the signed volume between the surface $z = e^{-x^2}$ and the triangle R in the xy-plane bounded by the x-axis, the line $x = 1$, and the line $y = x$.

Solutions

1. Find the signed volume between the surface $z = 1 + x + y$ and the region R in the xy-plane bounded by the graphs $x = 1, y = 0, y = x^2$.





Using sketch of R, shown on the left, notice that we can either say that

$$0 \leq x \leq 1 \text{ and } 0 \leq y \leq x^2$$

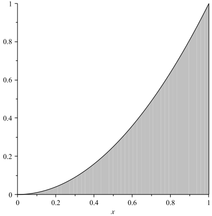
or

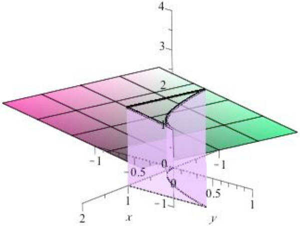
$$0 \leq y \leq 1 \text{ and } \sqrt{y} \leq x \leq 1.$$

$\boxed{\text{I will choose to use } 0 \leq x \leq 1 \text{ and } 0 \leq y \leq x^2}$

Solutions

1. Find the signed volume between the surface $z = 1 + x + y$ and the region R in the xy-plane bounded by the graphs $x = 1, y = 0, y = x^2$.





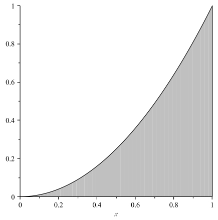
Choosing to use $0 \leq x \leq 1$ and $0 \leq y \leq x^2$

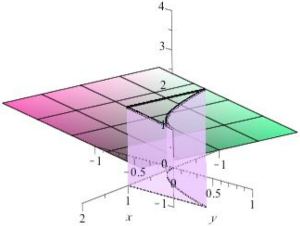
Thus by Fubini's Theorem,

$$V = \iint_R 1 + x + y \, dA = \int_0^1 \left(\int_0^{x^2} 1 + x + y \, dy \right) \, dx.$$

Solutions

1. Find the signed volume between the surface $z = 1 + x + y$ and the region R in the xy-plane bounded by the graphs $x = 1, y = 0, y = x^2$.



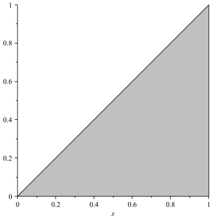


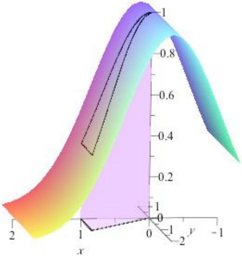
Thus by Fubini's Theorem,

$$
\begin{aligned}
V &= \iint_R 1 + x + y \, dA = \int_0^1 \left(\int_0^{x^2} 1 + x + y \, dy \right) dx \\
&= \int_0^1 \left(y + xy + \frac{y^2}{2} \Big|_0^{x^2} \right) dx \\
&= \int_0^1 x^2 + x^3 + \frac{1}{2} x^4 \, dx = \ldots = \frac{41}{60}
\end{aligned}
$$

Solutions

2. Find the volume below the surface $z = e^{-x^2}$ and above the triangle R in the xy-plane bounded by the x-axis, the line $x = 1$, and the line $y = x$.





Again, using sketch of region R on left, write the region in two ways :

$$0 \le x \le 1 \text{ and } 0 \le y \le x$$

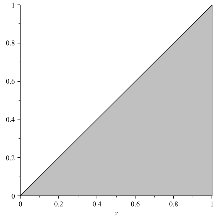
or

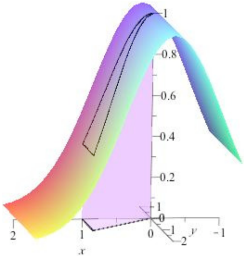
$$0 \le y \le 1 \text{ and } y \le x \le 1.$$

$$\boxed{\text{I will choose to use } 0 \le x \le 1 \text{ and } 0 \le y \le x}$$

Solutions

2. Find the volume below the surface $z = e^{-x^2}$ and above the triangle R in the xy-plane bounded by the x-axis, the line $x = 1$, and the line $y = x$.





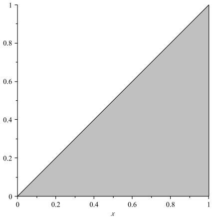
Choosing to use $0 \le x \le 1$ and $0 \le y \le x$

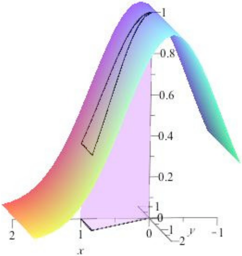
Thus by Fubini's Theorem,

$$V = \iint_R e^{-x^2}\, dA = \int_0^1 \left(\int_0^x e^{-x^2}\, dy \right)\, dx$$

Solutions

2. Find the volume below the surface $z = e^{-x^2}$ and above the triangle R in the xy-plane bounded by the x-axis, the line $x = 1$, and the line $y = x$.





Thus by Fubini's Theorem,

$$\begin{aligned} V &= \iint_R e^{-x^2}\, dA = \int_0^1 \left(\int_0^x e^{-x^2}\, dy \right) dx \\ &= \int_0^1 \left(ye^{-x^2} \Big|_0^x \right) dx = \int_0^1 xe^{-x^2}\, dx \\ &\overset{u-sub}{=} \ldots = -\frac{1}{2}\left(\frac{1}{e} - 1\right) = -\frac{1}{2e} + \frac{1}{2} \end{aligned}$$