Example: Intuitive notion of limits and continuity for $f : \mathbb{R}^2 \to \mathbb{R}$

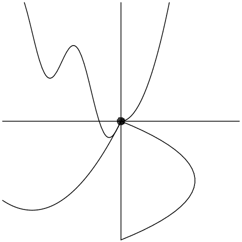
Let $f(x, y) = 1 - x^2 - y^2$. Find $\lim_{(x,y)\to(0,0)} f(x, y)$.

- If there's no problem at that point – you're not dividing by 0, you're not trying to plug 0 into natural log, you're not taking the square root of a negative number – in other words, if there's just no question as to whether f is continuous at the limiting point – then you can evaluate the limit by plugging the limiting point in.
- We can talk about informally about continuity even though we haven't formally defined continuity of a function in 2 dimensions yet.
- We have an intuitive idea of what "being discontinuous at a point means" - it means there's a problem at that point.
- This function does not have any problems anywhere - I can clearly evaluate f at any $(x, y) \in \mathbb{R}^2$.
- That is, f is continuous at $(0, 0)$
- And therefore $\lim_{(x,y)\to(0,0)} f(x, y) = f(0, 0) = 1$.

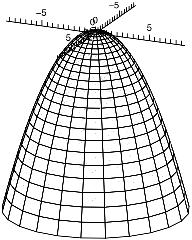
Intuitive Example, to Motivate Idea of Limits:

What that means:
No matter what path you look at that leads to $(0,0)$, on the surface, that path approaches $z = 1$.

A few possible paths



The surface $z = 1 - x^2 - y^2$



Recall: Conclusions:

- If $\lim_{(x,y)\to(a,b)} f(x,y) = L$ for some number L (that is, if the limit **does** exist), then f must approach L no matter what path (x,y) take to head toward (a,b).
- Consequently, if f approaches two different values along two different paths toward (a,b), the limit can not exist.
- Thus one way to show that a limit does not exist is to find two different paths in the xy-plane that both head toward (a,b), but which lead to different limits of $f(x,y)$.

Example from last time: Showing a Limit does not exist

Let $f(x,y) = \dfrac{3x^2}{x^2+y^2}$. What can we say about $\lim\limits_{(x,y)\to(0,0)} f(x,y)$?

- f is not defined at $(0,0)$, but the limit may still exist.
- Look at paths in the (x,y)-plane that go through $(0,0)$.
  - Every path we consider must go through $(0,0)$
  - Is there any point in looking at path $x = 3$? No! $(0,0)$ doesn't lie on it.
  - One path through $(0,0)$: $x = 0$, the y-axis.

$$\lim_{0,y)\to(0,0)} f(0,y) = \lim_{y\to 0} \frac{0}{y^2} = 0$$

  - Another path through $(0,0)$: $y = 0$, the x-axis.

$$\lim_{(x,0)\to(0,0)} f(x,0) = \lim_{x\to 0} \frac{3x^2}{x^2} = 3$$

- **Conclusion:** Because $f(x,y)$ approaches two different limits, $z = 3$ along the x and $z = 0$ along the y-axis, $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Daily WW #1:

Find the limit

$$\lim_{(x,y)\to(0,0)} \frac{(-7x+y)^2}{49x^2+y^2}$$

1. Along the x-axis: $y=0$

$$\lim_{(x,0)\to(0,0)} f(x,0) = \lim_{x\to 0} \frac{(-7x)^2}{49x^2} = 1$$

2. Along the y-axis: $x=0$

$$\lim_{(0,y)\to(0,0)} f(0,y) = \lim_{y\to 0} \frac{(y)^2}{y^2} = 1$$

3. Along the line $y=x$: $y=x$

$$\lim_{(x,x)\to(0,0)} f(x,x) = \lim_{x\to 0} \frac{(-7x+x)^2}{4x^2+x^2} = \frac{36}{50}$$

Because following different paths leads us to different limits,

$\displaystyle\lim_{(x,y)\to(0,0)} \frac{(-7x+y)^2}{49x^2+y^2}$ does not exist.

In Class Work

Consider $\lim_{(x,y)\to(1,0)} \dfrac{6x^2y - 12xy + 6y}{2(x-1)^4 + y^2}$.

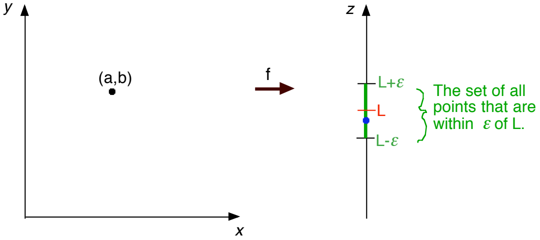
1. Find the limit along the path $y = 0$, $\lim_{(x,0)\to(1,0)} \dfrac{6x^2y - 12xy + 6y}{2(x-1)^4 + y^2}$.
2. Why would we *not* similarly consider the limit along the path $x = 0$?
3. Find the limit along the path $x = 1$.
4. Why would we *not* find the limit along the line $y = x$?
5. Find the limit along the path $y = x - 1$.
6. Find the limit along all lines through $(1, 0)$, $y = m(x - 1)$
7. Find the limit along the parabola $y = (x - 1)^2$

What can we conclude about $\lim_{(x,y)\to(1,0)} \dfrac{6x^2y - 12xy + 6y}{2(x-1)^4 + y^2}$?

Idea behind the definition of the multivariate limit:

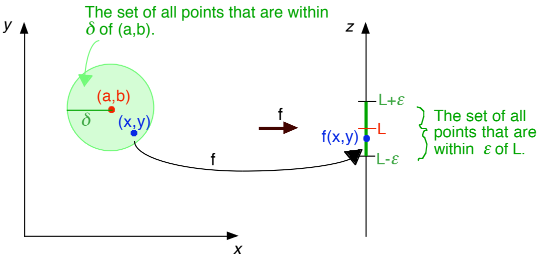
Given an element $(a, b) \in \mathbb{R}^2$, and a real number $L \in \mathbb{R}$. Does $\lim_{(x,y)\to(a,b)} f(x, y) = L$?

Let ϵ be an arbitrary positive real number. Mark off the set of all points within ϵ of L on the z-axis.



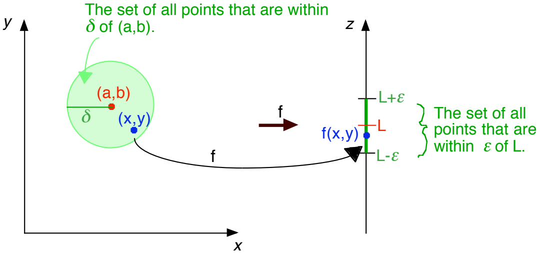
Idea behind the definition of the multivariate limit:

Does there exist a circle centered at (a, b), such that every point in that circle gets sent by f to the region around L that we marked off?



Idea behind the definition of the multivariate limit:

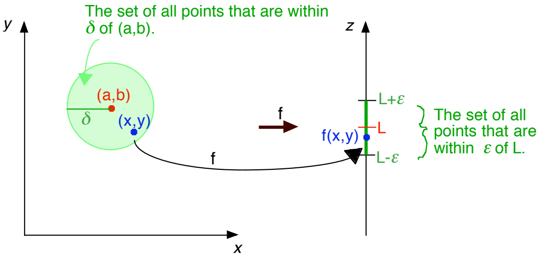
If, for *every* $\epsilon > 0$, there is a circle around (a, b) so that every point in the circle gets sent by f to the interval $(L - \epsilon, L + \epsilon)$, then we say that $\lim_{(x,y)\to(a,b)} f(x, y) = L$, because we can get arbitrarily close to L (ϵ close), just by choosing a small enough radius around (a, b).



Definition:

$\lim_{(x,y)\to(a,b)} f(x,y) = L$ if and only if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$d\big((x,y),(a,b)\big) < \delta \qquad \Longrightarrow \qquad |f(x,y) - L| < \epsilon$$



Solutions - In Class Work

Consider $\displaystyle\lim_{(x,y)\to(1,0)} \frac{6x^2y - 12xy + 6y}{2(x-1)^4 + y^2}$.

1. Find the limit along the path $y = 0$, $\displaystyle\lim_{(x,0)\to(1,0)} \frac{6x^2y - 12xy + 6y}{2(x-1)^4 + y^2}$.

$$\lim_{(x,0)\to(1,0)} \frac{6x^2y - 12xy + 6y}{2(x-1)^4 + y^2} = \lim_{x\to 1} \frac{0}{2(x-1)^4} = 0$$

2. Why would we *not* similarly consider the limit along the path $x = 0$?

   Because the point $(1, 0)$ does not lie on the line $x = 0$.

3. Find the limit along the path $x = 1$.

$$\lim_{(1,y)\to(1,0)} \frac{6x^2y - 12xy + 6y}{2(x-1)^4 + y^2} = \lim_{y\to 0} \frac{0}{y^2} = 0$$

Solutions - In Class Work - Continued

Consider $\displaystyle\lim_{(x,y)\to(1,0)} \frac{6x^2y-12xy+6y}{2(x-1)^4+y^2}$.

4. Why would we *not* find the limit along the line $y=x$?

   Again, because the point $(1,0)$ does not lie on the line $y=x$.

5. Find the limit along the path $y=x-1$.

$$\begin{aligned}
\lim_{(x,y)\to(1,0)} \frac{6x^2y-12xy+6y}{2(x-1)^4+y^2} &= \frac{6y(x^2-2x+1)}{2(x-1)^4+y^2} \\
&= \frac{6y(x-1)^2}{2(x-1)^4+y^2} \\
\lim_{(x,x-1)\to(1,0)} \frac{6x^2y-12xy+6y}{2(x-1)^4+y^2} &= \lim_{(x,x-1)\to(1,0)} \frac{6(x-1)^3}{2(x-1)^4+(x-1)^2} \\
&= \lim_{x\to1} \frac{6(x-1)}{2(x-1)^2+1} = 0
\end{aligned}$$

Solutions - In Class Work - Continued

Consider $\displaystyle\lim_{(x,y)\to(1,0)} \frac{6x^2y - 12xy + 6y}{2(x-1)^4 + y^2}$.

6. Find the limit along all lines through $(1,0)$, $y = m(x-1)$

$$\begin{aligned}
\lim_{(x,y)\to(1,0)} \frac{6x^2y - 12xy + 6y}{2(x-1)^4 + y^2} &= \frac{6y(x-1)^2}{2(x-1)^4 + y^2} \\
&= \lim_{(x,m(x-1))\to(1,0)} \frac{6m(x-1)^3}{2(x-1)^4 + m^2(x-1
\end{aligned}$$

$$= \lim_{x\to1} \frac{6m(x-1)}{2(x-1)^2 + m^2} = 0$$

Solutions - In Class Work - Continued

Consider $\displaystyle\lim_{(x,y)\to(1,0)} \frac{6x^2y-12xy+6y}{2(x-1)^4+y^2}$.

7. Find the limit along the parabola $y=(x-1)^2$

$$\begin{aligned}
\lim_{(x,y)\to(1,0)} \frac{6x^2y-12xy+6y}{2(x-1)^4+y^2} &= \frac{6y(x-1)^2}{2(x-1)^4+y^2} \\
&= \lim_{(x,(x-1)^2)\to(1,0)} \frac{6(x-1)^4}{2(x-1)^4+(x-1)^4} \\
&= \lim_{x\to1} \frac{6(x-1)^4}{3(x-1)^4} = 2
\end{aligned}$$

What can we conclude about $\displaystyle\lim_{(x,y)\to(1,0)} \frac{6x^2y-12xy+6y}{2(x-1)^4+y^2}$?

It took some looking, but it turns out that there are paths that lead to two different limits, and so the limit **does not exist**