Recall:

The function that gives the partial derivative of f with respect to x at any (x, y) is

$$\lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h} = \begin{cases} f_x(x, y) \\ \dfrac{\partial f}{\partial x}(x, y) \end{cases}.$$

$\dfrac{\partial f}{\partial x}(a, b)$ gives the rate that the function changes at the point $\big(a, b, f(a, b)\big)$ as we move along the curve on the surface with y fixed at b.

Recall:

Similarly, the function that gives the partial derivative of f with respect to y at any (x, y) is

$$\lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h} = \begin{cases} f_y(x, y) \\ \dfrac{\partial f}{\partial y}(x, y) \end{cases}.$$

$\dfrac{\partial f}{\partial x}(a, b)$ gives the rate that the function changes at the point $\big(a, b, f(a, b)\big)$ as we move along the curve on the surface with y fixed at b.

Example, continued:

Let $f(x, y) = x^2y + 3xy - y$.
We already found that

$$f_x(x, y) = 2xy + 3y \qquad f_y(x, y) = x^2 + 3x - 1$$

Find:

- $f_{xx}(x, y) = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right)$
- $f_{xy}(x, y) = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right)$
- $f_{yx}(x, y) = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right)$
- $f_{yy}(x, y) = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right)$

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- $f_{xx}(x, y) = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial}{\partial x}(2xy + 3y) = 2y$
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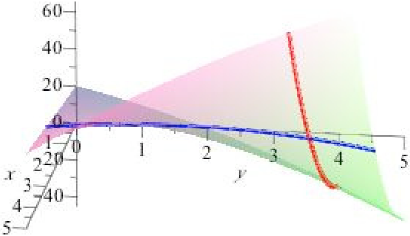
- $f_{yy}(x, y) = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial}{\partial y}(x^2 + 3x - 1) = 0$

In Class Work

Let $g(x, y) = x^2y + 4x - y^2 - 8y + xy + 20$.

1. Find $g_x(3, 4)$ and $g_y(3, 4)$. Look at the graph below and decide whether your results agree with what you're seeing.

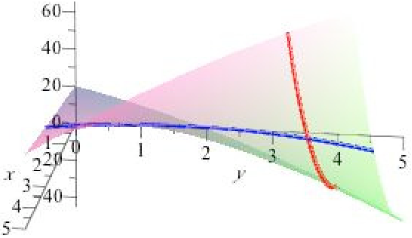


2. Find all points where $g_x(x, y)$ and $g_y(x, y)$ are both zero. What is the significance of these points?

3. Find g_{xy} and g_{yx}.

Solutions

1. Let $g(x, y) = x^2y + 4x + y^2 - 8y + xy + 20$. Find $g_x(3, 4)$ and $g_y(3, 4)$. Look at the graph below and decide whether your results agree with what you're seeing.



$$
\begin{aligned}
g_x(x, y) &= \frac{\partial}{\partial x} g(x, y) = 2xy + 4 + y \\
g_x(3, 4) &= 2(3)(4) + 4 + 4 = 32 \\
\\
g_y(x, y) &= x^2 - 2y - 8 + x \\
g_y(3, 4) &= 3^2 - 2(4) - 8 + 3 = -4
\end{aligned}
$$

- $g_x(3, 4) =$ slope of the curve $\langle x, 4, g(x, 4)\rangle$ at the point $(3, 4, g(3, 4))$. This curve is increasing rapidly there, which agrees with $g_x(3, 4) = 32$.
- $g_y(3, 4) =$ slope of the curve $\langle 3, y, g(3, y)\rangle$ at the point $(3, 4, g(3, 4))$. This curve is decreasing comparatively slowly there, which corresponds with $g_y(3, 4) = -4$.

Solutions

2. For $g(x,y) = x^2y + 4x + y^2 - 8y + xy + 20$, find all points where $g_x(x,y)$ and $g_y(x,y)$ are both zero.

Notice that if I just set the two partials equal to each other from the beginning, I'm trying to find **every** *point where they two partials are equal, and not using that they're in fact both 0.*

$$\begin{aligned} & g_x(x,y) = 0 \\ \Longrightarrow & 2xy + 4 + y = 0 \\ \Longrightarrow & (2x+1)y = -4 \\ \Longrightarrow & y = -\frac{4}{2x+1} \end{aligned}$$

$$\begin{aligned} & g_y(x,y) = 0 \\ \Longrightarrow & \; x^2 - 2y - 8 + x = 0 \\ \Longrightarrow & \; 2y = x^2 + x - 8 \end{aligned}$$

Solutions

2. (continued) For $g(x,y) = x^2y + 4x + y^2 - 8y + xy + 20$, find all points where $g_x(x,y)$ and $g_y(x,y)$ are both zero.

$$g_x(x,y) = 0 \implies y = -\frac{4}{2x+1} \qquad g_y(x,y) = 0 \implies 2y = x^2 + x - 8.$$

$$\begin{aligned}
g_x(x,y) = 0 = g_y(x,y) \quad &\implies \quad x^2 + x - 8 = 2y = 2\left(-\frac{4}{2x+1}\right) \\
&\implies \quad 2x^3 + x^2 + 2x^2 + x - 16x - 8 = -8 \\
&\implies \quad 2x^3 + 3x^2 - 15x = 0 \\
&\implies \quad x(2x^2 + 3x - 15) = 0 \\
&\implies \quad x = 0 \text{ or } x = \frac{-3 \pm \sqrt{9 - 120}}{4} \\
&\implies \quad x = 0 \\
&\implies \quad y = -4
\end{aligned}$$

Thus the only point where $g_x(x,y) = 0 = g_y(x,y)$ is the point $(0,-4)$.

Solutions

2. (continued) **What is the significance of the point** $(0, -4)$ **on the graph of** $g(x, y) = x^2y + 4x + y^2 - 8y + xy + 20$**?**

Solutions

2. (continued) **What is the significance of the point** $(0, -4)$ **on the graph of** $g(x, y) = x^2y + 4x + y^2 - 8y + xy + 20$**?**

- **If** a "nice" function $f(x, y)$ has a maximum or minimum value at (a, b), **then** we must have $f_x(a, b) = 0 = f_y(a, b)$.
- Thus for "nice" functions, local max's and min's **can only occur** at places where both $f_x(a, b) = 0 = f_y(a, b)$, so we have *critical points* – points that are *possible* min's and max's.

In this case, the point $(0, -4)$ is **a candidate** to be a max or a min, although without further investigation, we can't know for sure.