Goals:

Be able to :

1. determine whether a series $\sum a_k$ converges or diverges.
2. If it converges, find the limit (that is, the value of the series) exactly, if possible.
3. If it converges but we can't find the limit exactly, be able to approximate it.

How can we tell whether a series converges or diverges?

What we have so far:

1. Is it a geometric series? If so, and if $|r| \geq 1$, it diverges; if $|r| < 1$, it converges.
2. Does the Divergence Test apply? If $\lim_{k\to\infty} a_k \neq 0$, series diverges. If $\lim_{k\to\infty} a_k = 0$, test is inconclusive.

Recall: The Divergence Test

If the *sequence of terms* $\{a_k\}$ does not converge to zero, then the *series* $\sum_{k=0}^{\infty} a_k$ **must** diverge.

(Note: If the *sequence of terms* $\{a_k\}$ *does* converge to zero, then we have to investigate further.)

We got started proving *the equivalent statement*:

If $\sum_{k=0}^{\infty} a_k$ converges, then $\lim_{k\to\infty} a_k$ **must** be 0.

- **Suppose:** the series converges.

$$\begin{aligned}
\sum_{k=0}^{\infty} a_k \text{ converges} \quad &\overset{\text{def}}{\Longrightarrow} \quad \lim_{n\to\infty} S_n = L \text{ for some number } L \\
&\Longrightarrow \quad \lim_{n\to\infty} S_{n-1} \text{ also} = L \\
&\Longrightarrow \quad \lim_{n\to\infty} \left(S_n - S_{n-1} \right) = 0
\end{aligned}$$

In Class Work

1. Use the Comparison Test for Positive-Term Series to determine whether the following series converge or diverge, *OR* explain why none of the most obvious comparison choices help you decide.

   (a) $$\sum_{i=5}^{\infty} \frac{i-5}{i^4+8i}$$

   (b) $$\sum_{n=5}^{\infty} \frac{4^n+1}{2^n-n^2}$$

2. Use the Integral Test for Positive Term Series to determine whether the following series converge or diverge, *OR* explain why it does not apply.

   (a) $$\sum_{j=3}^{\infty} \frac{18j^2}{3j^3+1}$$

   (b) $$\sum_{k=1}^{\infty} \frac{k}{e^{k^2}}$$

Solutions:

Use the Comparison Test for Positive-Term Series to determine whether the following series converge or diverge, *OR* explain why none of the most obvious comparison choices help you decide.

1(a) $\sum_{i=5}^{\infty} \frac{i-5}{i^4+8i}$ Every term in this series is positive.

For all values of i, $i-5 \leq i$, and $i^4+8i \geq i^4 \quad \Rightarrow \quad \frac{1}{i^4+8i} \leq \frac{1}{i^4}$.

$$\Rightarrow \frac{i-5}{i^4+8i} \leq \frac{i-5}{i^4} \leq \frac{i}{i^4} = \frac{1}{i^3}$$

$$\sum_{i=5}^{\infty} \frac{i-5}{i^4+8i} \leq \sum_{i=5}^{\infty} \frac{1}{i^3}$$

Since the series on the right is a p-series with $p=3$, and since $p>1$, the series on the right converges.

Thus the series on the left converges as well.

Solutions:

Use the Comparison Test for Positive-Term Series to determine whether the following series converge or diverge, *OR* explain why none of the most obvious comparison choices help you decide.

1(b) $\displaystyle\sum_{n=5}^{\infty} \frac{4^n+1}{2^n-n^2}$ Every term in this series is positive.

For all values of n, $4^n+1 \geq 4^n$, and $2^n - n^2 \leq 2^n \quad \Rightarrow \quad \dfrac{1}{2^n-n^2} \geq \dfrac{1}{2^n}$.

$$\Rightarrow \frac{4^n+1}{2^n-n^2} \quad \geq \quad \frac{4^n+1}{2^n} \geq \frac{4^n}{2^n} = 2^n$$

$$\sum_{n=5}^{\infty} \frac{4^n+1}{2^n-n^2} \quad \geq \quad \sum_{n=5}^{\infty} 2^n$$

Since the series on the right is a geometric series with $r = 2$, and since $|r| > 1$, the series on the right diverges.
Thus the series on the left diverges as well.

Solutions:

Use the Integral Test for Positive Term Series to determine whether the following series converge or diverge, *OR* explain why it does not apply.

2(a) $\displaystyle\sum_{j=3}^{\infty} \frac{18j^2}{3j^3+1}$

The graph of $f(x) = \dfrac{18x^2}{3x^3+1}$ shows that this function is continuous, positive, and decreasing, so the integral test applies.

Integral Test $\Longrightarrow \displaystyle\sum_{j=3}^{\infty} \frac{18j^2}{3j^3+1}$ will do whatever $\displaystyle\int_3^{\infty} \frac{18x^2}{3x^3+1}\,dx$ does.

$$\int_3^{\infty} \frac{18x^2}{3x^3+1}\,dx = 2\int_{28}^{\infty} \frac{1}{u}\,du = 2\lim_{R\to\infty}\int_{28}^{R} \frac{1}{u}\,du$$

We know this integral diverges, so the series diverges as well.

Solutions:

Use the Integral Test for Positive Term Series to determine whether the following series converge or diverge, *OR* explain why it does not apply.

2(b) $\displaystyle\sum_{k=1}^{\infty} \frac{k}{e^{k^2}}$

Again, the graph shows that the function is continuous, positive, and decreasing from about $x = 1$ on, so the integral test applies.

Integral Test $\Longrightarrow \displaystyle\sum_{k=1}^{\infty} \frac{k}{e^{k^2}}$ will do whatever $\displaystyle\int_1^{\infty} xe^{-x^2}\,dx$ does.

$$\int_1^{\infty} xe^{-x^2}\,dx = \int_{-1}^{-\infty} -\frac{1}{2}e^u\,du = -\frac{1}{2}\lim_{R\to\infty} e^u\Big|_{-1}^{-R} = \frac{1}{2e}.$$

Since this integral converges, so does the series (although it does not converge to $\frac{1}{2e}$)