Fun Fact: The Set Up

The set of a game show has three closed doors. Behind one door is a car; behind the other two are goats. The contestant does not know where the car is, but the host does.

- The contestant picks a door.
- The host opens one of the two *remaining* doors, one he **knows** doesn't hide the car, showing one of the two goats. (If the contestant has chosen the correct door, the host is equally likely to open either of the two remaining doors.)
- After the host has shown a goat, the contestant is given the option to switch doors.

Question: What is the probability that the contestant will win the car if she stays with her first choice? Does that probability change if she changes to the remaining door?

Fun Fact

The contestant should switch doors after being shown the goat.

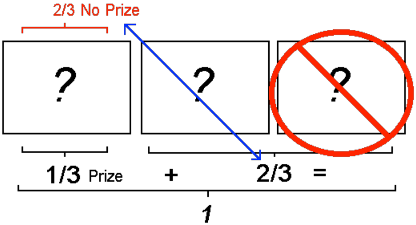
The probability of winning the car if she doesn't switch doors is only $1/3$, while the probability of winning the car if she does switch doors is $2/3$!

Fun Fact

Claim: The probability of winning the car if she doesn't switch doors is only 1/3, while the probability of winning the car if she does switch doors is 2/3!

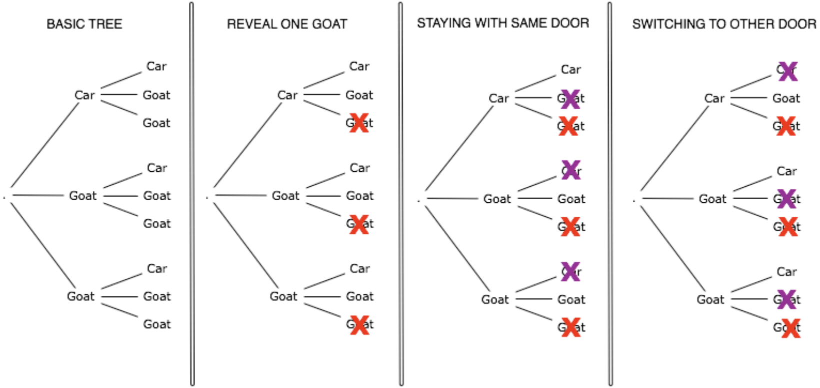
Why?

- When the contestant first chooses a door, the car is equally likely to be behind any of the 3 doors. That is ...
- P(car behind chosen door)=1/3
  $\Rightarrow$ P(NOT behind chosen door)=2/3
- Host opens one of the two doors the contestant *didn't* choose.
- But the probability that the car wasn't behind the 1st door is still 2/3!



Fun Fact

If the previous reasoning doesn't convince you, here is a list of all the possibilities if the contestant doesn't switch choices, and if she does:



Staying with the same door leads to you winning 1 out of 3 times; switching doors leads to you winning 2 out of 3 times.

Goals:

Be able to :

1. Determine whether a series $\sum a_k$ converges or diverges.
   - Geometric Series?
   - Divergence Test?
   - Integral Test?
   - Comparison Test?
2. If it converges, find the limit (that is, the value of the series) exactly, if possible.
   - Geometric series only (so far)
3. If it converges but we can't find the limit exactly, be able to approximate it.

Recall: The Integral Test

If $a(x)$ is a function that is continuous, positive, and decreasing on $[1, \infty)$, and $\{a_k\}$ is the sequence defined by $a_k = a(k)$ for every integer $k \geq 1$, then

$$\sum_{k=1}^{\infty} a_k \text{ and } \int_1^{\infty} a(x)\, dx$$

either both converge or both diverge.

Used the integral test to show:

$$\sum_{k=1}^{\infty} \frac{1}{k^p} \begin{cases} \text{converges when } p > 1 \\ \text{diverges when } p \leq 1. \end{cases}$$

Series of the form $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^p}$ are called ***p*-series.**

(Remember: the divergent p-series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ = the **Harmonic Series**).

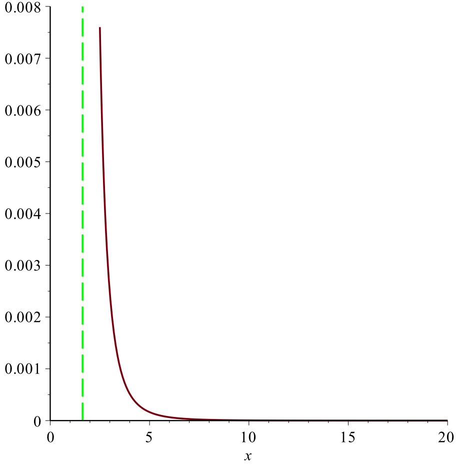
Recall:

Let $a(x)$ be a continuous, positive, decreasing function on $[1, \infty)$, and define $a_k = a(k)$ for all whole numbers k.

When looking at why the Integral Test for Positive Term Series is true, we treated $\sum a_k$ as left or right sums with $\Delta x = 1$, and came up with the following string of inequalities:

$$\int_2^\infty a(x)\, dx \leq \sum_{k=2}^{\infty} a_k \leq \int_1^\infty a(x)\, dx \leq \sum_{k=1}^{\infty} a_k$$

Graph of $a(x) = \dfrac{2x^3}{(2x^4 - 14)^2}$ on $[2.5, \infty)$



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In Class Work

Let $S = \sum_{k=2}^{\infty} \frac{k}{e^{k^2}}$.

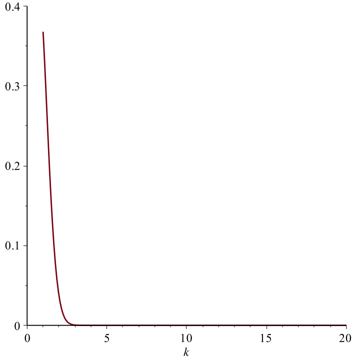
1. Verify that the integral test applies.
2. Show that S converges.
3. Find lower and upper limits for the value of S.
4. Find an integer N so that the partial sum $S_N = \sum_{k=2}^{N} \frac{k}{e^{k^2}}$ is within 0.00001 of S.

Solutions:

Let $S = \sum_{k=2}^{\infty} \frac{k}{e^{k^2}}$.

1. Verify that the integral test applies.

   Graph of $\frac{x}{e^{x^2}}$ on $[1, \infty)$



$a(x) = \frac{x}{e^{x^2}}$ is continuous, positive, and decreasing on $[2, \infty)$.

Thus the integral test applies.

Solutions:

Let $S = \displaystyle\sum_{k=2}^{\infty} \frac{k}{e^{k^2}}$.

2. Show that S converges.

   Since the integral test, S converges if $\displaystyle\int_2^\infty \frac{x}{e^{x^2}}\,dx$ converges, and S diverges if $\displaystyle\int_2^\infty \frac{x}{e^{x^2}}\,dx$ diverges.

$$\begin{aligned}
\int_2^\infty \frac{x}{e^{x^2}}\,dx &= \lim_{B\to\infty}\int_2^B \frac{x}{e^{x^2}}\,dx \\
&= \lim_{B\to\infty}\int_{x=2}^{x=B} -\frac{1}{2}e^u\,du \text{ (with } u=-x^2) \\
&= \lim_{B\to\infty} -\frac{1}{2}e^{-x^2}\Big|_2^B = \lim_{B\to\infty} -\frac{1}{2e^{B^2}} + \frac{1}{2e^4} \text{ converges!}
\end{aligned}$$

Thus S converges as well (but not to the same value)

Solutions:

Let $S = \displaystyle\sum_{k=2}^{\infty} \frac{k}{e^{k^2}}$.

3. Find lower and upper limits for the value of S.

$$\text{Integral test} \quad \Longrightarrow \quad \int_2^{\infty} a(x)\,dx \le \sum_{k=2}^{\infty} a_k \le \int_1^{\infty} a(x)\,dx$$

$$\Longrightarrow \quad \int_2^{\infty} xe^{-x^2}\,dx \le S \le \int_1^{\infty} xe^{-x^2}\,dx$$

Since we found that $\displaystyle\int_2^{\infty} xe^{-x^2}\,dx = \frac{1}{2e^4}$,

$$\frac{1}{2e^4} \le S \le \lim_{B\to\infty} -\frac{1}{2}e^{-x^2}\Big|_1^B$$

$$\frac{1}{2e} \le S \le \frac{1}{2e}$$

$$0.009 \le S \le 0.184$$

Solutions:

Let $S = \sum_{k=2}^{\infty} \frac{k}{e^{k^2}}$.

4. Find an integer N so that the partial sum $S_N = \sum_{k=2}^{N} \frac{k}{e^{k^2}}$ is within 0.00001 of S.

$$\underbrace{\sum_{k=2}^{\infty} \frac{k}{e^{k^2}}}_{S} = \underbrace{\sum_{k=2}^{N} \frac{k}{e^{k^2}}}_{S_N} + \underbrace{\sum_{k=N+1}^{\infty} \frac{k}{e^{k^2}}}_{R_N}$$

If find N so that $R_N \leq 0.00001$, then S_N is within 0.00001 of S.

Know $R_N \leq \int_N^{\infty} \frac{x}{e^{x^2}}\, dx$.

So if we find N so that $\int_N^{\infty} \frac{x}{e^{x^2}}\, dx \leq 0.00001$, then $R_N \leq 0.00001$, and so S_N will be within 0.00001 of S.

Solutions:

Let $S = \sum_{k=2}^{\infty} \frac{k}{e^{k^2}}$.

4. (continued) Find an integer N so that the partial sum $S_N = \sum_{k=2}^{N} \frac{k}{e^{k^2}}$ is within 0.00001 of S. Want to find N so that $\int_N^{\infty} \frac{x}{e^{x^2}}\,dx \le 0.00001$.

$$\begin{aligned}
\int_N^{\infty} \frac{x}{e^{x^2}}\,dx \le 0.00001 \quad &\Rightarrow \quad \lim_{B\to\infty} -\frac{1}{2}e^{-x^2}\Big|_N^B \le \frac{1}{100000} \\
&\Rightarrow \quad \frac{1}{2e^{N^2}} \le \frac{1}{100000} \Rightarrow 50000 \le e^{N^2} \\
&\Rightarrow \quad \ln(50000) \le N^2 \Rightarrow N \ge \sqrt{\ln(50000)} \\
&\Rightarrow \quad N \ge 3.29
\end{aligned}$$

Thus $\sum_{k=2}^{4} \frac{k}{e^{k^2}}$ is within 0.00001 of $\sum_{k=2}^{\infty} \frac{k}{e^{k^2}}$