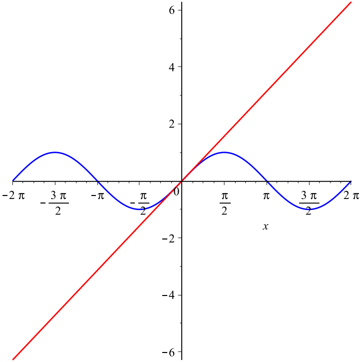
Goals:

Be able to :

1. Determine whether a series $\sum a_k$ converges or diverges.
   - Geometric Series?
   - Divergence Test?
   - Integral Test?
   - Comparison Test?
2. If it converges, find the limit (that is, the value of the series) exactly, if possible.
   - Geometric series only (so far)
3. If it converges but we can't find the limit exactly, be able to approximate it.

Motivation for Taylor Series:

Example:



Let $f(x) = \sin(x)$.
Then $f'(x) = \cos(x)$.
Line tangent to $f(x)$ at $x_0 = 0$:

- goes thru the point $(0, \sin(0)) = (0, 0)$
- has slope $m = \cos(0) = 1$

$P_1(x) = 1(x - 1) + 0$
$\Rightarrow P_1(x) = x$

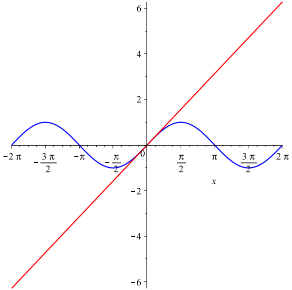
In general: if $f(x)$ diff'ble at x_0 and $P_1(x)$ is line tangent to $f(x)$ x_0,

Tangent line P_1 is the only line thru $\big(x_0, f(x_0)\big)$ with slope $f'(x_0)$.
That is, the tangent line is **the** line $P_1(x)$ with

$$P_1(x_0) = f(x_0) \qquad \text{and} \qquad P_1'(x_0) = f'(x_0)$$

Motivation for Taylor Series:

Example:



The tangent line at $x = x_0$, $P_1(x)$, is **the** line with

$$P_1(x_0) \quad = \quad f(x_0)$$

and

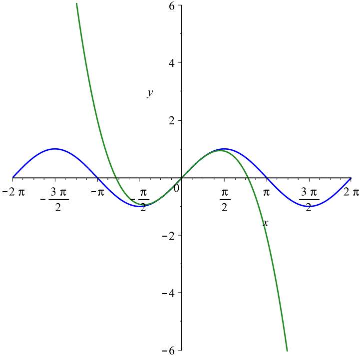
$$P_1'(x_0) \quad = \quad f'(x_0)$$

Because the tangent line *agrees with* $f(x)$ at x_0, and the slope of the tangent line *agrees with* the slope of $f(x)$ at x_0, the tangent line does a good job of approximating $f(x)$ *near* x_0.

Note: The tangent line and the function are equal **at** x_0; **near** x_0, we can use the tangent line to estimate the function.

Motivation for Taylor Series

Example:



Let $f(x) = \sin(x)$.

Let $P_3(x) = -\dfrac{x^3}{6} + x$

$\Rightarrow P_3'(x) = -\frac{x^2}{2} + 1$

$\Rightarrow P_3''(x) = -x$

$\Rightarrow P_3'''(x) = -1$

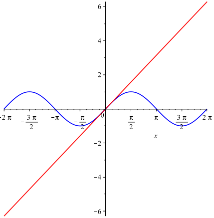
Thus (check!) $P_3(x)$ is the cubic with

$$\begin{aligned} P_3(0) &= 0 = & f(0) \\ P_3'(0) &= 1 = & f'(0) \\ P_3''(0) &= 0 = & f''(0) \\ P_3'''(0) &= -1 = & f'''(0) \end{aligned}$$

$P_3(x)$ does an even better job at approximating $f(x)$ near x_0

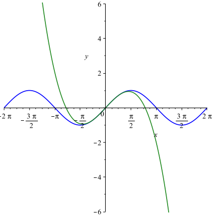
Motivating Taylor Series

When $f(x) = \sin(x)$,



1st degree poly $P_1(x) = x$ has

$$\begin{aligned} P_1(0) &= f(0) \\ P_1'(0) &= f'(0) \end{aligned}$$



3rd degree poly $P_3(x) = -\frac{x^3}{6} + x$ has

$$\begin{aligned} P_3(0) &= f(0) \\ P_3'(0) &= f'(0) \\ P_3''(0) &= f''(0) \\ P_3'''(0) &= f'''(0) \end{aligned}$$

What would make a 7th degree poly approximate $\sin(x)$ well near $x = 0$?

Motivating Taylor Series

Question: What would make a 7th degree poly approximate $\sin(x)$ well near $x = 0$?

Finding a 7th degree polynomial $P_7(x)$ that has

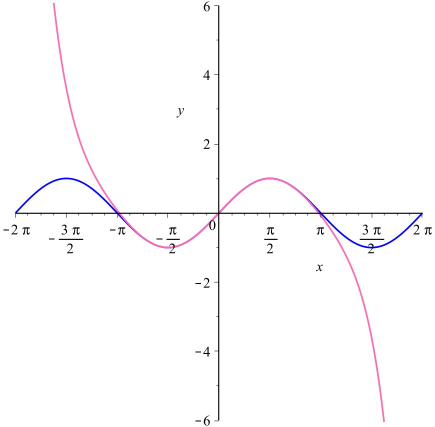
$$\begin{aligned} P_7(0) &= f(0) \\ P_7'(0) &= f'(0) \\ &\vdots \\ P_7^{(6)}(0) &= f^{(6)}(0) \\ P_7^{(7)}(0) &= f^{7)}(0) \end{aligned}$$

Motivating Taylor Series

Using techniques you'll be learning soon, we can find that

$$P_7(x) = -\frac{x^7}{7!} + \frac{x^5}{5!} - \frac{x^3}{3!} + x$$

is such a 7th degree polynomial.



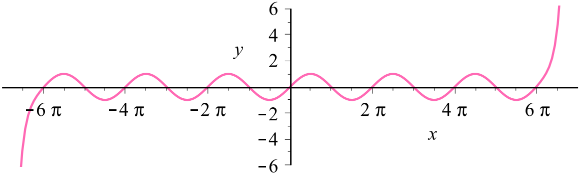
Motivating Taylor Series

We can use these *Taylor Polynomial* approximations to approximate

- The value of a more complicated function at a particular point
- The derivative of a more complicated function at a point
- The integral of a more complicated function on an interval

The more derivatives *at* $x = 0$ we match (the higher n is), the better $P_n(x)$ does at approximating $\sin(x)$ *away* from $x = 0$.

The following is a graph of $P_{49}(x)$:



Question: What would make the idea of Taylor polynomials even more accurate?

nth degree Taylor Polynomial for $f(x)$ near $x_0 = 0$:

Let $f(x)$ be an arbitrary function, differentiable at x_0.

Create an nth degree polynomial P_n so that

$$P_n(0) = f(0) \qquad P_n'(0) = f'(0) \qquad P_n''(0) = f''(0) \quad \ldots \quad P_n^{(n)}(0) = f^{(n)}(0)$$

Let $P_n(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_3x^3 + a_2x^2 + a_1x + a_0$.

k	$f^{(k)}(x)$	$f^{(k)}(0)$	$P_n^{(k)}(x)$	$P_n^{(k)}(0)$
0	$f(x)$	$f(0)$	$P_n(x) = a_nx^n + \ldots a_2x^2 + a_1x + a_0$	$P_n(0) = a_0$
1	$f'(x)$	$f'(0)$	$P_n'(x) = na_nx^{n-1} + \ldots 2a_2x + a_1$	$P_n'(0) = a_1$
2	$f''(x)$	$f''(0)$	$P_n''(x) = n(n-1)a_nx^{n-2} + \ldots + 2!a_2$	$P_n''(0) = 2!a_2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$f^{(n)}(x)$	$f^{(n)}(0)$	$P_n^{(n)}(x) = n!a_n$	$P_n^{(n)}(0) = n!a_n$

Solving for the a_k gives us

$$a_0 = f(0) \qquad a_1 = f'(0) \qquad a_2 = \frac{f''(0)}{2!} \qquad \ldots \qquad a_n = \frac{f^{(n)}(0)}{n!}$$

In Class Work

Find the 6th degree Taylor Polynomial for $\cos(x)$ based at $x = 0$, and use it to approximate $\cos(0.1)$.

Solutions:

Find the 6th degree Taylor Polynomial for $\cos(x)$ based at $x = 0$, and use it to approximate $\cos(0.1)$.

$$P_6(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$$

$$\cos(0.1) \approx P_6(0.1) = 1 - \frac{0.1^2}{2!} + \frac{0.1^4}{4!} - \frac{0.1^6}{6!} = 0.9950041653$$

Compare this to what your calculator gives for $\cos(0.1)$:

$$\cos(0.1) \approx 0.9950041653$$

That's an extremely good approximation!