Daily WeBWorK Problem 3

Compute $P_2(x)$ for $y = e^x$, based at $x_0 = 0.8$. Compute the error incurred by using P_2 to approximate e^x at $x = 0.7$.

k	$f^{(k)}(x)$	$a_k = \frac{f^{(k)}(0.8)}{k!}$
0	$f(x) = e^x$	$a_0 = \frac{e^{0.8}}{0!} = e^{0.8}$
1	$f'(x) = e^x$	$a_1 = \frac{e^{0.8}}{1!} = e^{0.8}$
2	$f''(x) = e^x$	$a_2 = \frac{e^{0.8}}{2!} = \frac{e^{0.8}}{2}$

$$\Rightarrow P_2(x) = \overbrace{e^{0.8}}^{a_0} + \overbrace{e^{0.8}}^{a_1}(x-0.8) + \overbrace{\frac{e^{0.8}}{2}}^{a_2}(x-0.8)^2$$

$$\Rightarrow e^{0.7} \approx P_2(0.7) = e^{0.8}\left(1 + (0.7-0.8) + \frac{1}{2}(0.7-0.8)^2\right)$$

$$\approx 2.014114540$$

$$e^{0.7} \overset{\text{Maple}}{=} 2.013752707 \Rightarrow \text{Error} = \left|e^{0.7} - P_2(0.7)\right| \approx 0.000361$$

Taylor Series We've Seen

- $\sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$
- $\sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{(x-\frac{\pi}{2})^{2k}}{(2k)!} = 1 - \frac{(x-\frac{\pi}{2})^2}{2!} + \frac{(x-\frac{\pi}{2})^4}{4!} + \cdots$
- $\cos(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$
- $\ln(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(x-1)^k}{k} = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} + \cdots$
- $\frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$

In Class Work

1. Find the Taylor Series for e^x based at $x = 0$ (that is, find the Maclaurin Series for e^x).
2. Use known Taylor Series to find the Taylor series for
   (a) $\dfrac{1}{1+x^2}$
   (b) $\arctan(x)$
3. Find a series that gives $\int_0^1 e^{-x^2}$ exactly.

Taylor Series You Should Know

- $\sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$
- $\cos(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$
- $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$
- $\ln(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(x-1)^k}{k} = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} + \cdots$
- $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots$
- $\arctan(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \cdots$

Notation and Vocabulary for $\mathbb{R}^n$:

- One dimension = real-number line = $\mathbb{R}$.
- 2-dimensional space = plane $=\mathbb{R}^2$. We'll call it "2-space" sometimes.
- 3-dimensional space= space$=\mathbb{R}^3$. We'll call it "3-space".

  $\vdots$
- In general, n dimensions$=\mathbb{R}^n=$ n-space.

3-dimensional space is the highest number of dimensions we can readily graph.

Solutions

1. Find the Taylor Series for e^x about $c = 0$

k	$f^{(k)}(x)$	$a_k = \frac{f^{(k)}(0)}{k!}$
0	e^x	$a_0 = \frac{e^0}{0!} = 1$
1	e^x	$a_1 = \frac{e^0}{1!} = 1$
2	e^x	$a_2 = \frac{e^0}{2!} = \frac{1}{2!}$
3	e^x	$a_3 = \frac{e^0}{3!} = \frac{1}{3!}$
$\vdots$	$\vdots$	$\vdots$
k	e^x	$a_k = \frac{e^0}{k!} = \frac{1}{k!}$

The Taylor Series for e^x is

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\ &= \sum_{k=0}^{\infty} \frac{x^k}{k!} \end{aligned}$$

Solutions

2(a) Find the Taylor Series for $\dfrac{1}{1+x^2}$:

Since the Taylor Series for $\dfrac{1}{1+x}$ based at $x=0$ is $\sum_{k=0}^{\infty}(-1)^k x^k = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$, the Taylor Series for $\dfrac{1}{1+x^2}$ based at $x=0$ is

$$\sum_{k=0}^{\infty}(-1)^k (x^2)^k = 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \cdots$$

2(b) Find the Taylor Series for $\arctan(x)$

Using the result from 2(a):

$$\begin{aligned}\arctan(x) &= \int \frac{1}{1+x^2}\,dx = \int \left(1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \cdots\right) dx \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \cdots\end{aligned}$$

Solutions

3. Find a series that gives $\int_0^1 e^{-x^2}\,dx$ exactly.

$$\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\
\Rightarrow e^{-x^2} &= 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots \\
\Rightarrow \int_0^1 e^{-x^2}\,dx &= \int_0^1 \left(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots\right) dx \\
&= \left[x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 4!} + \cdots\right]_0^1 \\
&= 1 - \frac{1}{3} + \frac{1}{5 \cdot 2!} - \frac{1}{7 \cdot 4!} + \cdots \\
&= \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1) \cdot k!}
\end{aligned}$$