Goal:

- We've seen that in some cases, adding up an infinite number of numbers – an infinite sum– makes sense:

$$\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k \overset{\text{seems to}}{=} \quad 1$$

$$0 < \sum_{n=2}^{\infty}\frac{1}{n^2} \leq \quad 1$$

- **Goal:** Work toward formalizing the idea of an **infinite sum**.
- In order to do this, first must discuss infinite lists of numbers – **Sequences**.

Sequences:

- **Formal Definition:** A **sequence of real numbers** $\{a_k\}$ is a function whose domain is the set of integers, starting with some integer n_0 (often 0 or 1). The **terms** of the sequence are the individual outputs of the function: for each positive integer $k \geq n_0$, the output a_k is called the **kth term** of the sequence.

  Example The function $a(k) = 2^k$, for $k = 0, 1, 2, 3, \ldots$ defines the sequence

$$\{a_k\}_{k=0}^{\infty} = \{1, 2, 4, 8, 16, \ldots\}$$

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  In this example, since $1 = a(0)$, 1 is referred to as the **0th term** and denoted a_0. Since $2 = a(1)$, 2 is the **first term**, a_1, etc. We call 2^k the **general term** a_k.

Question:

Find a symbolic expression for the general term a_k of the sequence

$$\{0, 3, 6, 9, 12, 15, \ldots\}$$

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Answer: $\{0, 3, 6, 9, 12, 15, \ldots\} = \{3k\}_{k=0}^{\infty}$

Question We Can Ask About Sequences:

What are the terms of the sequence doing in the long run? Are they increasing without bound? Bouncing around? Approaching some specific value?

- In other words,

  Do the terms of the sequence **converge** to a limiting value L, or do they diverge, either by approaching $\pm\infty$ or by not approaching anything at all?

- To determine convergence/divergence, use the same basic idea behind convergence and divergence as we always have – **but** . . .
- . . . be aware difference is that our domain is just integers rather than all real numbers.

In Class Work

In each case: find the first three terms of the sequence (beginning with whatever index you're given), then decide whether the following sequences converge or diverge, and if the sequence converges, find the limit.

1. $\left\{ \dfrac{\sqrt{n}}{\ln(n)} \right\}_{n=2}^{\infty}$

2. $\left\{ \dfrac{100m^2 + 200}{.01m^3 - 500m^2} \right\}_{m=2}^{\infty}$

3. $\left\{ \dfrac{5e^{6n} + 100n}{10e^{6n} + n^{100}} \right\}_{n=0}^{\infty}$

Intro to Partial Sums:

Consider a sequence $\{a_k\}_{k=0}^{\infty} = \{a_0, a_1, a_2, a_3, \ldots\}$.

- *One* thing we can do with a sequence is what we've been doing:
  - look to see what the terms do as $k \to \infty$.
- **Not** the *only* thing we can do with it.
- Another thing we can do: form a new sequence!

Solutions to In Class Work:

Do the following sequences converge or diverge?

If the sequence converges, find the limit.

1. $\left\{ \dfrac{\sqrt{n}}{\ln(n)} \right\}_{n=2}^{\infty}$

First couple terms:

$$\left\{ \frac{\sqrt{2}}{\ln(2)}, \frac{\sqrt{3}}{\ln(3)}, \frac{\sqrt{4}}{\ln(4)}, \dots \right\} \approx \{2.04, 1.58, 1.44, \dots\}$$

Does it converge or diverge?

$$\begin{aligned}
\lim_{n\to\infty} \frac{\sqrt{n}}{\ln(n)} &= \frac{\infty}{\infty} \text{ Indeterminate form! } \quad \text{Use l'Hôpital's Rule} \\
&= \lim_{n\to\infty} \frac{\frac{1}{2\sqrt{n}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{2\sqrt{n}} = \lim_{n\to\infty} \frac{\sqrt{n}}{2} \text{ Not indeterminate} \\
&= \infty
\end{aligned}$$

Thus this sequence diverges (to ∞).

Solutions:

2. $\left\{ \dfrac{100m^2 + 200}{.01m^3 - 500m^2} \right\}_{m=2}^{\infty}$

First couple terms:

$$\left\{ \frac{600}{-1999.92}, \frac{1100}{-4499.73}, \frac{1800}{-7999.36}, \dots \right\} \approx \{-.300, -.244, -.225, \dots\}$$

$$\begin{aligned}
\lim_{m\to\infty} \frac{100m^2 + 200}{.01m^3 - 500m^2} &= \frac{\infty}{\infty}. \text{ Indeterminate form! } \quad \text{Use l'Hôpital's} \\
&= \lim_{m\to\infty} \frac{200m}{.03m^2 - 1000m} = \frac{\infty}{\infty}. \text{ Use l'Hôpital's} \\
&= \lim_{m\to\infty} \frac{200}{.03m - 1000} = \frac{200}{\infty} \text{ Not indeterminate} \\
&= 0
\end{aligned}$$

This sequence converges, to 0.

Solutions:

3. $\left\{ \dfrac{5e^{6n} + 100n}{10e^{6n} + n^{100}} \right\}_{n=0}^{\infty}$

First couple terms:

$$\left\{ \frac{5}{10}, \frac{5e^6 + 100}{10e^6 + 1}, \frac{5e^{12} + 200}{10e^{12} + 2^{100}}, \ldots \right\} \approx \{0.5, 0.525, 6.42 * 10^{-25}, 6.37 * 10^{-40}, 8.24 * 10^{-50}, 6.77 * 10^{-57}, \ldots\}$$

Looks as if it's going to approach 0!

Except... The leaps in order of magnitude are getting smaller, so maybe it's slowing down. Maybe it will approach some very very very small non-zero number?
Let's see!

Solutions:

3. (continued)

$$\begin{aligned}
\lim_{n\to\infty} \frac{5e^{6n}+100n}{10e^{6n}+n^{100}} &= \frac{\infty}{\infty} \text{l'Hôpital's!} \\
&= \lim_{n\to\infty} \frac{6\cdot 5e^{6n}+100}{6\cdot 10e^{6n}+100n^{99}} = \frac{\infty}{\infty}\text{l'Hôpital's} \\
&= \lim_{n\to\infty} \frac{6\cdot 6\cdot 5e^{6n}}{6\cdot 6\cdot 10e^{6n}+100\cdot 99n^{98}} = \frac{\infty}{\infty} \\
&\vdots \\
&= \lim_{n\to\infty} \frac{6^{100}5e^{6n}}{6^{100}10e^{6n}+100!} = \frac{\infty}{\infty} \\
&= \lim_{n\to\infty} \frac{6^{101}5e^{6n}}{6^{101}10e^{6n}} = \frac{1}{2}
\end{aligned}$$

Thus this sequence eventually converges, and not to where we expected – not to 0, or even to a very small non-zero number. It converges to $\frac{1}{2}$ – its starting term!