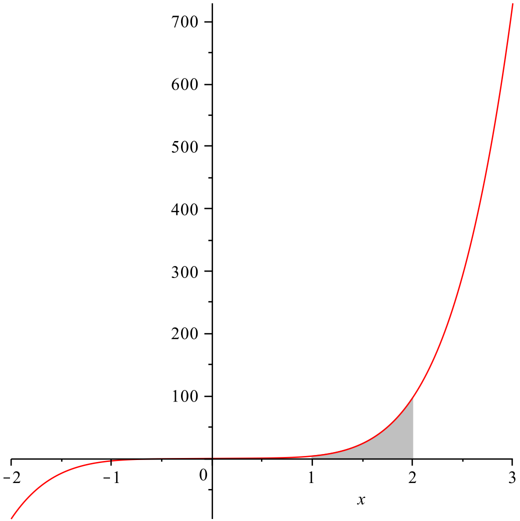
Where We're Going:

Continue Our Quick Review of the Basics of Integration

- The Fundamental Theorem of Calculus
- Antidifferentiation
- Signed Area; Area btwn 2 curves

The signed area between $3x^5 + \sin(x)$ and the x-axis from $x = 1$ to $x = 2$



The Fundamental Theorem of Calculus

If f is continuous on $[a, b]$ and F is any antiderivative of f, then

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

Question:

- If a function $f(x)$ is continuous, must it have an antiderivative?
- If yes: Must there be a *formula* for the antiderivative?

Question:

- If a function $f(x)$ is continuous, must it have an antiderivative?
- If yes: Must there be a *formula* for the antiderivative?

The Second Fundamental Theorem of Calculus

If f is continuous on $[a, b]$ and we define $A_f(x) = \int_a^x f(t)\,dt$ for all $x \in [a, b]$, then

(a) A_f is continuous on $[a, b]$ and differentiable on (a, b)

(b) A_f is an antiderivative of f; that is, $A_f'(x) = f(x)$.

Note: The text uses the notation $F(x)$: I prefer A_f as it reminds us that this is the area from a to x under f

Questions:

What are definite integrals? Indefinite integrals?

What are the differences between a definite integral and an indefinite integral? What does each one represent?

Definite Integrals vs Indefinite Integrals

- The definite integral $\int_a^b f(x)\,dx$ is *defined* to be the signed area between the curve f and the x-axis from $x = a$ to $x = b$.

- The FTC (parts 1 and 2) tell us of the connection between this signed area and antiderivatives.

- Because hitherto there has been no good notation for antiderivatives, we take advantage of this connection to create some:

  We use $\int f(x)\,dx$ to represent the family of all antiderivatives of f,

  and we call $\int f(x)\,dx$ the indefinite integral

One challenge with antidifferentiation

Consider the two products xe^{x^2} and xe^x.

(a) Which differentiation rule would you use to verify that an antiderivative of xe^{x^2} is $\frac{1}{2}e^{x^2}$?

(b) Which differentiation rule would you use to verify that an antiderivative of xe^x is $e^x(x-1)$?

Why do your answers to (a) and (b) make it unlikely that we will find a general product rule for antidifferentiation?

In Class Work

1. Find two antiderivatives for $p(x) = 3x^5 + 7x^4 - \frac{3}{x} + \frac{11}{x^2}$ (by hand).
2. Find the function $v(t)$ satisfying the conditions that $v'(t) = 2e^t - 3\cos(3t)$ and $v(0) = -3$.
3. Find the area between $f(x) = x^2$ and $g(x) = 8 - x^2$. [Note that I am asking for area, which should be positive, not signed area, which can be positive or negative.]
4. Suppose that $f(x) = x^2 - 3e^{x^2} + 4$ and $A_f(x) = \int_{-5}^{x} f(t)\,dt$.
   (a) Is the graph of $f(x)$ increasing or decreasing at $x = -2$?
   (b) Is the graph of $f(x)$ concave up or concave down at $x = -2$?
   (c) Is the graph of $A_f(x)$ increasing or decreasing at $x = -2$?
   (d) Is the graph of $A_f(x)$ concave up or concave down at $x = -2$?

Solutions

1. Find two antiderivative for $p(x) = 3x^5 + 7x^4 - \frac{3}{x} + \frac{11}{x^2}$ by hand.

   One antiderivative: $P(x) = \frac{3}{6}x^6 + \frac{7}{5}x^5 - 3\ln|x| - \frac{11}{x}$

   No need for $+C$. Creates family of all *antiderivatives.*

   Adding any constant to $P(x)$ will give me a second antiderivative. For example, a second antiderivative is

$$Q(x) = \frac{3}{6}x^6 + \frac{7}{5}x^5 - 3\ln|x| - \frac{11}{x} + e.$$

Solutions

2. Find the function $v(t)$ satisfying the conditions that $v'(t) = 2e^t - 3\cos(3t)$ and $v(0) = -3$.

Need to find: Specific antiderivative of $v'(t)$ that has $v(0) = 3$.

First, find any antiderivative:

- If you remember integration by substitution, use it.
- If you don't, guess, check and modify until you find an antiderivative. (We'll review substitution one day next week)

The family of antiderivatives is

$$v(t) = 2e^t - \sin(3t) + C$$

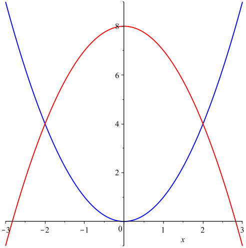
Need to find: The value of C that makes $v(0) = -3$:

$$-3 = v(0) = 2e^0 - \sin(0) + C = 2 + C \Longrightarrow C = -5.$$

Thus $v(t) = 2e^t - \sin(3t) - 5$.

Solutions:

3. Find the area between $f(x) = x^2$ and $g(x) = 8 - x^2$.



By solving $x^2 = 8 - x^2$ for x, we find that the intersection points are $x = \pm 2$.

$$\begin{aligned}
\text{Area} &= \text{signed area under } g(x) \text{ - signed area under } f(x) \\
&= \int_{-2}^{2} (8 - x^2) - x^2 \, dx = \int_{-2}^{2} 8 - 2x^2 \, dx \\
&= 8x - \frac{2}{3}x^3 \Big|_{-2}^{2} = \left(16 - \frac{16}{3}\right) - \left(-16 + \frac{16}{3}\right) \\
&= 32 - \frac{32}{3}
\end{aligned}$$

Solutions:

4. Suppose that $f(x) = x^2 - 3e^{x^2} + 4$, and $A_f(x) = \int_{-5}^{x} f(t)\,dt$.

   (a) Is the graph of $f(x)$ increasing or decreasing at $x = -2$?

   $$f'(x) = 2x - 6xe^{x^2} \Rightarrow f'(-2) = -4 + 12e^4 > 0 \Rightarrow f \text{ is } \uparrow \text{ at } x = -2$$

   (b) Is the graph of $f(x)$ concave up or concave down at $x = -2$?

   $$f''(x) = 2 - 12x^2e^{x^2} - 6e^{x^2} \Rightarrow f''(-2) = 2 - (48 + 6)e^4 < 0.$$

   f is concave down at $x = -2$.

   (c) Is the graph of $A_f(x)$ increasing or decreasing at $x = -2$?

   Since A_f is an antiderivative of f, A_f' is f.

   $$A_f'(x) = f(x) = x^2 - 3e^{x^2} + 4 \Rightarrow A_f'(-2) = 4 - 3e^4 + 4 < 0.$$

   A_f is decreasing at $x = -2$.

   (d) Is the graph of $A_f(x)$ concave up or concave down at $x = -2$?

   Since $A_f''(x) = f'(x)$,

   $$A_f''(-2) = f'(-2) = -4 + 12e^4 > 0 \Rightarrow A_f \text{ concave up at } x = -2$$