Daily WeBWorK

3.

$$\int \left(\frac{4}{5x} + 2\right)\, dx = \int \frac{4}{5}\frac{1}{x} + 2\, dx = \frac{4}{5}\ln|x| + 2x + C$$

5. Let $f(x) = 1 - x^2$. Over the interval $[0, 5]$, find

   (a) The signed area between f and the x-axis

$$\text{Signed Area} = \int_0^5 1 - x^2\, dx = \left(x - \frac{1}{3}x^3\right)\Big|_0^5 = \left(5 - \frac{5^3}{3}\right) - 0$$

   (b) The absolute area between f and the x-axis
   Absolute area: all area is positive. Separate the signed area which is already positive from the area which is negative.
   $1 - x^2$ is positive on $[0, 1]$, negative on$[1, 5]$, and so the signed area from $[1, 5]$ will be negative. Thus $-\int_1^5 1 - x^2\, dx$ will be positive.

$$\begin{aligned}\text{Absolute Area} &= Y = \int_0^1 1 - x^2\, dx - \int_1^5 1 - x^2\, dx \\ &= \left[\left(1 - \frac{1}{3}\right)\right] - \left[\left(5 - \frac{5^3}{3}\right) - \left(1 - \frac{1}{3}\right)\right]\end{aligned}$$

Where We're Going

Goal: To be able to antidifferentiate as many functions as possible.

$\displaystyle\int \frac{1}{1+x^2}\,dx$, $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx$, and $\displaystyle\int x\cos(x^2)\,dx$ – all very basic looking antiderivatives. And yet they are each non-trivial, in their own way.

- (Review) Inverse Trig Functions
- (Review?) Their derivatives
- (Review) Integration by Substitution

Review - Inverse Functions

- f and g are **inverse** functions if for all x in the domains of f and g,

$$\begin{aligned} f\big(g(x)\big) &= x \\ \text{and } g\big(f(x)\big) &= x \end{aligned}$$

That is, f and g are inverses if g undoes f and f undoes g.

- **Example:** e^x and $\ln(x)$ are inverse functions:

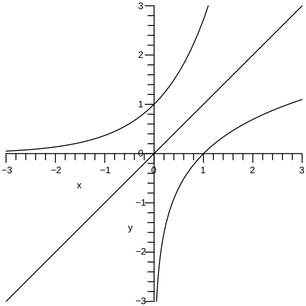
$$\text{For all } x, e^{\ln(x)} = x \qquad \ln(e^x) = x$$

For instance,

$$f\big(g(1)\big) = e^{\ln(1)} = e^0 = 1 \qquad g\big(f(1)\big) = \ln(e^1) = 1$$

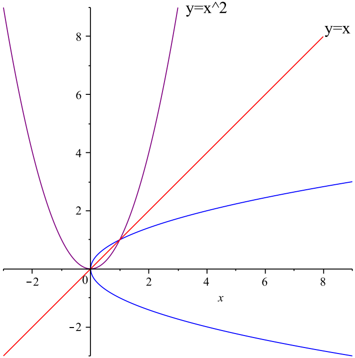
Review - Inverse Functions

- The graphs of inverse functions are reflections across the line $y = x$, as illustrated with the graphs of e^x, $\ln(x)$, and $y = x$ shown below:



Review - Inverse Functions

- **Recall:** Not every function has an inverse.
- **Example:** $f(x) = x^2$

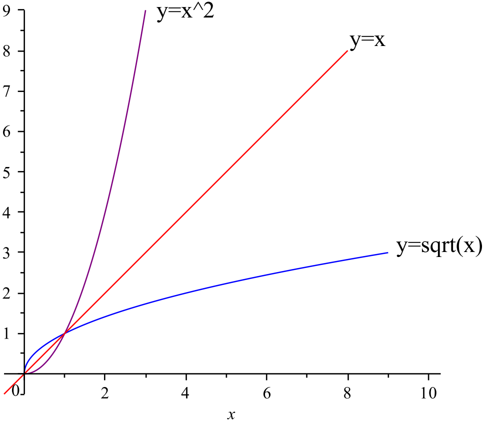


The reflection of $y = x^2$ across the line $y = x$ results in a graph which is not a function (there are two outputs for most inputs).

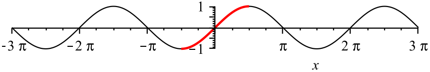
$y = x^2$ is not *one-to-one* – For most outputs (y values), you could pick either one of two inputs (x-values) to give it.

Review - Inverse Functions

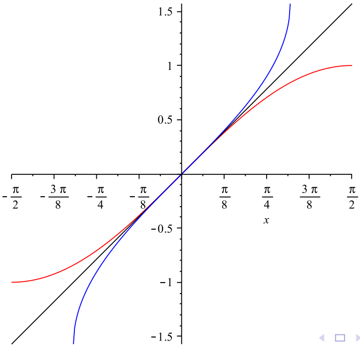
- If we restrict the domain of $f(x) = x^2$ to $x \geq 0$, then it is invertible and $g(x) = \sqrt{x}$ is the inverse.



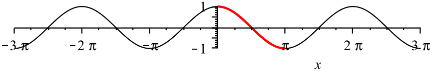
Review - Defining arcsin(x)



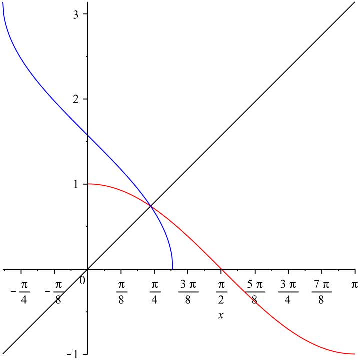
For any x in $[-1, 1]$, define $\arcsin(x)$ to be the unique y-value in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ where $x = \sin(y)$. $\arcsin(x) = \square \Leftrightarrow \sin(\square) = x$.



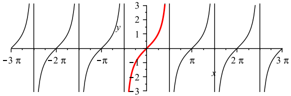
Defining $\arccos(x)$



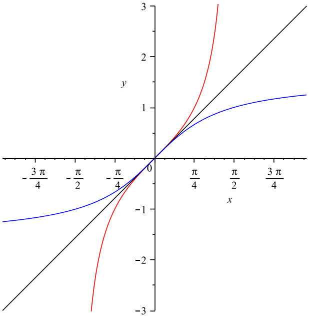
For any x in $[-1, 1]$, define $\arccos(x)$ to be the unique y-value in the interval $[0, \pi]$ where $x = \cos(y)$. $\arccos(x) = \square \Leftrightarrow \cos(\square) = x$.



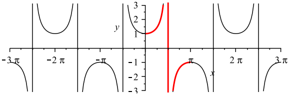
Defining arctan(x)



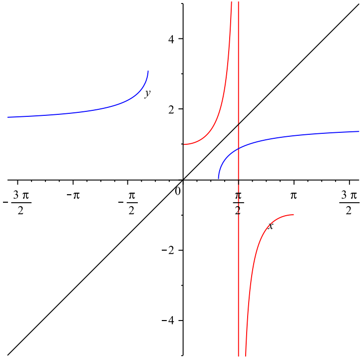
For any x in $(-\infty, \infty)$, define arctan(x) to be the unique y-value in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ where $x = \tan(y)$. $\arctan(x) = \square \Leftrightarrow \tan(\square) = x$.



Defining arcsec(x)



For any x in $(-\infty, \infty)$, define arcsec(x) to be the unique y-value in the interval $(0, \pi)$ where $x = \sec(y)$. $\text{arcsec}(x) = \square \Leftrightarrow \sec(\square) = x$.



Question from your reading:

Integration by substitution attempts to undo one of the techniques of differentiation – which one?

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The chain rule

Example:

Differentiate $f(x) = \cos(x^3)$

$f(x)$ is a composition.
Let $u = x^3$, $g(u) = \cos(u)$

$$\frac{d}{dx}(f(x)) = \frac{dg}{du}\frac{du}{dx} = \big(-\sin(u)\big)\big(3x^2\big) = -3x^2\sin(x^3)$$

Antidiff $h(x) = -3x^2\sin(x^3)$

$h(x)$ is a product; one piece is a composition.
Let $u = x^3$. Then $\frac{du}{dx} = 3x^2$

$$h(x) = -\sin(u)\frac{du}{dx}$$

Treat $-\sin(u)$ as $\dfrac{dg}{du}$.
$\Rightarrow g(u) = \cos(u)$.

$$H(x) = \cos(x^3)$$

Concept behind substitution

$$\frac{d}{dx}f(u(x)) = f'(u(x))\frac{du}{dx}, \text{ so } \int f'(u(x))u'(x)\ dx = f(u(x)) + C$$

Without all the x's:

$$\frac{d}{dx}f(u) = f'(u)\frac{du}{dx}, \text{ so } \int f'(u)\frac{du}{dx}\ dx = f(u) + C$$

Notation: In this change of variables, we rewrite $\dfrac{du}{dx}\ dx$ as du

$$\frac{d}{dx}f(u) = f'(u)\frac{du}{dx}, \text{ so } \int f'(u)\ du = f(u) + C$$

Integration by Substitution:

1. Find a composition.
2. Let $u =$ the inside function.
3. Find $\frac{du}{dx}$.
4. Find du: $du = \frac{du}{dx} \cdot dx$.

   If $du = \frac{du}{dx}\, dx$ more or less appears as part of the product (give or take a multiplicative constant), then substitution may work.
5. Substitute in du and u where they go. Can not omit du, and du can not be inside any function (or the denominator of any fraction). No x's can remain at the end of the substitution – all must be replaced by equivalent expressions in u (and one du).
6. Antidifferentiate in u: The antiderivative of $\int f'(u)\, du$ is just $f(u)$.
7. Resubstitute: We now must substitute back in for the original $u(x)$.
8. Check your results by differentiating them!

In Class Work

1. Find the following derivatives. Don't worry about algebraic simplifications.
   (a) $\frac{d}{dx}\left(\arcsin(x^2)\right)$
   (b) $\frac{d}{dx}\left(e^x \arctan(4x)\right)$
2. Find the following indefinite or definite integrals, and *check your answers by differentiating*
   (a) $\int \frac{1}{\sqrt{1-x}}\,dx$
   (b) $\int \frac{4}{\sqrt{1-4x^2}}\,dx$
   (c) $\int x\sin(\pi x^2)\,dx$
   (d) $\int_1^3 \frac{x}{1+x^2}\,dx$
   (e) $\int \frac{x}{1+x^4}\,dx$

Solutions

1. Find the following derivatives. Don't worry about algebraic simplifications.

   (a) $\frac{d}{dx}\left(\arcsin(x^2)\right) = \frac{1}{\sqrt{1-(x^2)^2}} \cdot 2x = \frac{2x}{\sqrt{1-x^4}}$

   (b) $$\frac{d}{dx}\left(e^x \arctan(4x)\right) = e^x\left(\frac{1}{1+(4x)^2} \cdot 4\right) + e^x \arctan(4x)$$
   $$= \frac{4e^x}{1+16x^2} + e^x \arctan(4x)$$

Solutions:

2(a). Find the following indefinite or definite integrals; check answers.

(a) $\displaystyle\int \frac{1}{\sqrt{1-x}}\,dx$

- Substitute:
  - Composition: $\sqrt{1-x}$.
  - Let $u = 1 - x$.
  - Differentiating $u \Rightarrow \dfrac{du}{dx} = -1 \Rightarrow du = -1\,dx$
  - Substitute
    $$\int \frac{1}{\sqrt{1-x}}\,dx = \int \frac{1}{\sqrt{1-x}} \cdot -1 \cdot -1\,dx = \int \frac{1}{\sqrt{u}} \cdot -1\,du = -\int u^{-1/2}\,du.$$
- Antidifferentiate in u:
  $$-\int u^{-1/2}\,du = -\frac{1}{1/2}u^{1/2} + C = -2\sqrt{u} + C.$$
- Resubstitute:
  $$\int \frac{1}{\sqrt{1-x}}\,dx = -2\sqrt{1-x} + C.$$

Check: $\dfrac{d}{dx}\left(-2\sqrt{1-x} + C\right) = -2 \cdot \dfrac{1}{2}(1-x)^{-1/2} \cdot (-1) + 0$

Solutions:

2(b) $\displaystyle\int \frac{4}{\sqrt{1-4x^2}}\,dx$

- Substitute:
  - Composition: $\sqrt{1-4x^2} = \sqrt{1-(2x)^2}$
  - Let $u = 2x$.
  - Differentiating $u \Rightarrow \dfrac{du}{dx} = 2 \Rightarrow du = 2\,dx$
  - Substitute:

$$\int \frac{4}{\sqrt{1-4x^2}}\,dx = \int \frac{2}{\sqrt{1-(2x)^2}} \cdot 2\,dx$$

$$= \int \frac{2}{\sqrt{1-u^2}} \cdot du = 2\int \frac{1}{\sqrt{1-u^2}}\,du.$$

- Antidifferentiate in u:

$$2\int \frac{1}{\sqrt{1-u^2}}\,du = 2\arcsin(u) + C.$$

- Resubstitute: $\displaystyle\int \frac{4}{\sqrt{1-4x^2}}\,dx = 2\arcsin(2x) + C.$

Check: $\dfrac{d}{dx}\big(2\arcsin(2x) + C\big) = 2 \cdot \dfrac{1}{\sqrt{1-(2x)^2}} \cdot 2 + 0$

Solutions:

2(c) $\int x\sin(\pi x^2)\,dx$

- Substitute:
  - Composition: $\sin(\pi x^2)$
  - Let $u = \pi x^2$.
  - Differentiating $u \Rightarrow \frac{du}{dx} = 2\pi x \Rightarrow du = 2\pi x\,dx$
  - Substitute:

$$\begin{aligned}\int x\sin(\pi x^2)\,dx &= \int \sin(\pi x^2)\cdot \frac{1}{2\pi}\cdot 2\pi x\,dx \\ &= \int \sin(u)\cdot\frac{1}{2\pi}\,du = \frac{1}{2\pi}\int \sin(u)\,du.\end{aligned}$$

- Antidifferentiate in u:

$$\frac{1}{2\pi}\int \sin(u)\,du = \frac{1}{2\pi}\cdot -\cos(u) + C.$$

- Resubstitute: $\int x\sin(\pi x^2)\,dx = -\frac{1}{2\pi}\cos(\pi x^2) + C.$

Check: $\frac{d}{dx}\left(-\frac{1}{2\pi}\cos(\pi x^2) + C\right) = -\frac{1}{2\pi}\cdot -\sin(\pi x^2)\cdot 2\pi x$

Solutions

2(d) $\displaystyle\int_1^3 \frac{x}{1+x^2}\,dx$

- Substitute:
  - Composition: $\dfrac{x}{1+x^2} = x(1+x^2)^{-1}$.
  - Let $u = 1 + x^2$
  - Differentiating $u \Rightarrow \dfrac{du}{dx} = 2x \Rightarrow du = 2x\,dx$
  - Substitute:

$$\begin{aligned}\int_1^3 \frac{x}{1+x^2}\,dx &= \int_{x=1}^{x=3} \frac{1}{1+x^2}\cdot x\,dx = \int_{u=2}^{u=10} \frac{1}{1+x^2}\cdot\frac{1}{2}\cdot 2x\,dx \\ &= \int_2^{10} \frac{1}{u}\cdot\frac{1}{2}\,du = \frac{1}{2}\int_2^{10}\frac{1}{u}\,du.\end{aligned}$$

- Antidifferentiate in u:

$$\frac{1}{2}\int_2^{10}\frac{1}{u}\,du = \frac{1}{2}\ln|u|\Big|_{10}^{2} = \frac{1}{2}\left(\ln|10| - \ln|2|\right) = \frac{1}{2}\ln\left(\frac{10}{2}\right) = \ln(5^{1/2})$$

Check:

$$\frac{d}{dx}\left(\frac{1}{2}\ln|1+x^2|\right) = \frac{1}{2}\cdot\frac{1}{1+x^2}\cdot 2x$$

Solutions

2(e) $\displaystyle\int \frac{x}{1+x^4}\,dx$

- Substitute:
  - Composition: $\frac{x}{1+x^4} = x(1+x^4)^{-1} = x(1+(x^2)^2)^{-1}$.
  - Letting $u = x^4$ won't work. In that case, $du = 4x^3\,dx$, and there is no x^3 term present.
  - Instead, let $u = x^2$.
  - Differentiating $u \Rightarrow \dfrac{du}{dx} = 2x \Rightarrow du = 2x\,dx$
  - Replacing x^2 with u and $x\,dx$ with $\frac{1}{2}\,du$:

$$\int \frac{x}{1+x^4}\,dx = \int \frac{1}{1+(x^2)^2}\cdot\frac{1}{2}\cdot 2x\,dx = \int \frac{1}{1+u^2}\cdot\frac{1}{2}\,du = \frac{1}{2}\int\frac{1}{1+u^2}\,du.$$

- Antidifferentiate in u:

$$\frac{1}{2}\int\frac{1}{1+u^2}\,du = \frac{1}{2}\arctan(u) + C.$$

- Resubstitute:

$$\int \frac{x}{1+x^4}\,dx = \frac{1}{2}\arctan(x^2) + C.$$

Remember to check by differentiating!