Error Bounds for Trapezoidal and Midpoint Sums

Suppose that f is integrable and either always concave up or always concave down on $[a, b]$.
Let n be any positive integer, and let

$$E_{T_n} = \underbrace{\left|\mathcal{I} - T_n\right|}_{\text{error from using } T_n \text{ to approx } \mathcal{I}} \qquad E_{M_n} = \underbrace{\left|\mathcal{I} - M_n\right|}_{\text{error from using } M_n \text{ to approx } \mathcal{I}}$$

Then

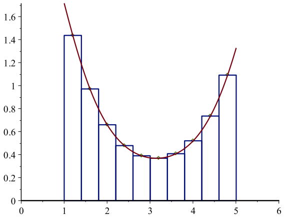
$$\underbrace{E_{T_n}, E_{M_n}}_{\text{either exact error}} \leq \underbrace{\left|T_n - M_n\right|}_{\text{an error bound}}.$$

If we further suppose that f'' is bounded on $[a, b]$ – that is, that there exists some positive real number M such that $\left|f''(x)\right| \leq M$ for all x in $[a, b]$, then

$$E_{T_n} \leq \underbrace{\frac{M(b-a)^3}{12n^2}}_{\text{larger error bound}} \qquad E_{M_n} \leq \underbrace{\frac{M(b-a)^3}{24n^2}}_{\text{larger error bound}}$$

Example:

Let $\mathcal{I} = \int_1^5 e^{\cos(x)}\, dx$.



Notice that $e^{\cos(x)}$ is concave up on $[1, 5]$

It turns out that

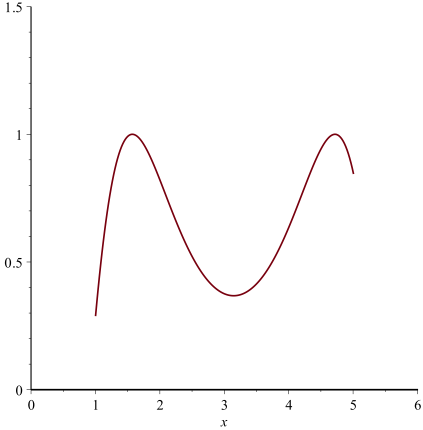
$$M_{10} \approx 2.823611872.$$

Thus

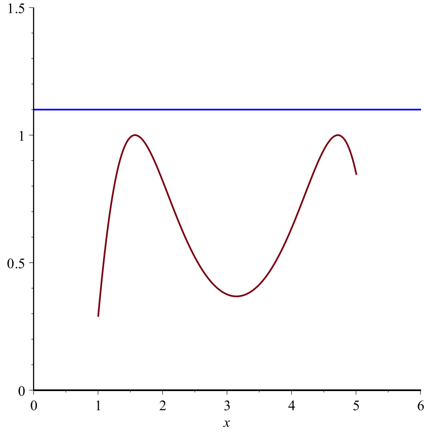
$$\int_0^{10} e^{\cos(x)}\, dx \approx 2.823611872.$$

But how close **is** $\mathcal{I}$ to 2.823611872?

To find M, look at graph of $f''(x)$ on $[1, 5]$:



To find M, look at graph of $f''(x)$ on $[1, 5]$:



Because $0 \leq f''(x) \leq 1.1$, can use $M = 1.1$

Recall: Formal Definition of a Definite Integral

Let f be a function defined on an interval $[a, b]$.

The **definite integral** of f from $x = a$ to $x = b$ is defined to be the number

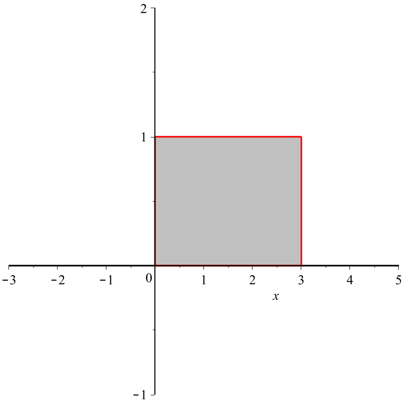
$$\int_a^b f(x)\, dx \stackrel{\text{def}}{=} \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$$

if this limit exists, where

- $\Delta x = \frac{b-a}{n}$
- $[x_0, x_1, \ldots, x_{k-1}, x_k, \ldots, x_{n-1}, x_n]$ is a partition of $[a, b]$ with $x_k - x_{k-1} = \Delta x$ for all $k = 1 \ldots n$
- x_k^* is any choice of point in the subinterval $[x_{k-1}, x_k]$

Question 1:

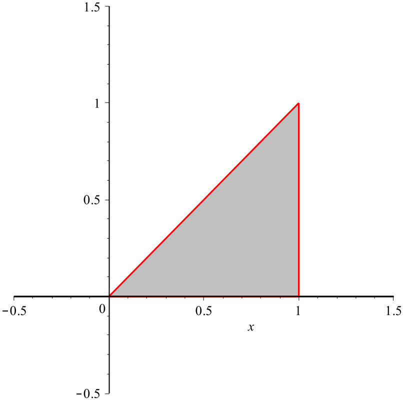
Let R be the rectangle formed by the x-axis, the y-axis, and the lines $y = 1$ and $x = 3$.



What shape is the solid formed when R is rotated about the x-axis?

Question 2:

Let T be the triangle formed by the lines $y = x$, $x = 1$ and the x-axis.



What shape is the solid formed when T is rotated about the x-axis?

Plan for Finding Volume with Slices

To find the volume of an object:

- We have seen that if the area of a cross-section=area at x is $= A(x)$ then

$$\text{Volume} = \int_a^b A(x)\,dx$$

- To find volume, we thus need to first figure out what $A(x)$ is.