WeBWorK, Part (a)

Approximate $\int_1^9 \ln(x)\,dx$ using a midpoint sum with $n = 12$. Then find an error bound for your approximation.

- Finding M_{12}:

$$\begin{aligned} \Delta x &= \frac{9-1}{12} = \frac{2}{3} \\ \Rightarrow \text{Partition} &= \{1, \frac{5}{3}, \frac{7}{3}, \frac{9}{3}, \frac{11}{3}, \frac{13}{3}, \frac{15}{3}, \frac{17}{3}, \frac{19}{3}, \frac{21}{3}, \frac{23}{3}, \frac{25}{3}, \frac{27}{3} = 9\} \end{aligned}$$

Thus

$$\begin{aligned} M_{12} &= \frac{2}{3}\Bigg[\ln\left(\frac{4}{3}\right) + \ln\left(\frac{6}{3}\right) + \ln\left(\frac{8}{3}\right) + \ln\left(\frac{10}{3}\right) + \ln\left(\frac{12}{3}\right) \\ &\quad + \ln\left(\frac{14}{3}\right) + \ln\left(\frac{16}{3}\right) + \ln\left(\frac{18}{3}\right) + \ln\left(\frac{20}{3}\right) + \ln\left(\frac{22}{3}\right) \\ &\quad + \ln\left(\frac{24}{3}\right) + \ln\left(\frac{26}{3}\right)\Bigg] \end{aligned}$$

WeBWorK, Part (b)

Error in using M_{12} to approximate $\displaystyle\int_1^9 \ln(x)\, dx$:

$$E_{M_{12}} \leq \frac{M(9-1)^3}{24(12)^2},$$

where $\left|\dfrac{d^2}{dx^2}\big(\ln(x)\big)\right| \leq M$ for all $x \in [1,9]$.

$$\frac{d}{dx}\big(\ln(x)\big) = \frac{1}{x} \Rightarrow \frac{d^2}{dx^2}\big(\ln(x)\big) = \frac{-1}{x^2}$$

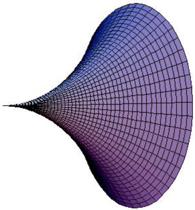
On $[1,9]$, the biggest $\left|\dfrac{-1}{x^2}\right|$ ever gets occurs at $x = 1$, so we can use $M = 1$.

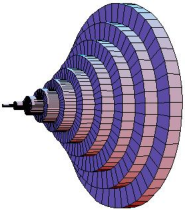
Thus

$$E_{M_{12}} \leq \frac{1(9-1)^3}{24(12)^2}.$$

Recall:

If a solid occurs btwn $x = a$ and $x = b$, and the slices are perpendicular to the x-axis, we can slice it, approximate the volume of each slice, and add the estimated-slice-volumes.



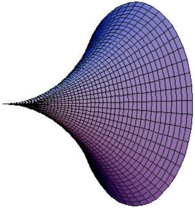


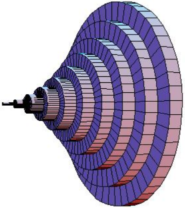
Treat the slices as having straight sides by choosing one x within each slice, x_i^*, and using the cross-section through that one point to define the shape of the slice throughout its entire thickness.
Thus the volume of the ith slice is (thickness) $\times$ (area at x_i^*)

$$\text{Volume} \approx \sum_{i=1}^{n} A(x_i^*)\Delta x$$

Recall:

If a solid occurs btwn $x = a$ and $x = b$, and the slices are perpendicular to the x-axis,, we can slice it, approximate the volume of each slice, and add the estimated-slice-volumes.





The more slices we take, the better the approximation. In fact,

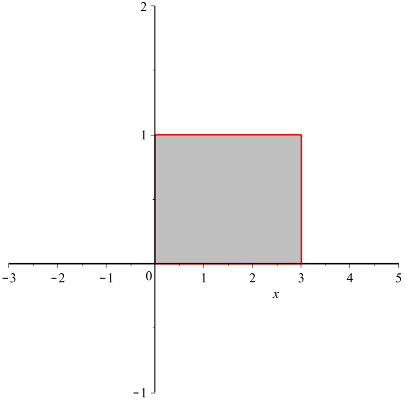
$$\text{Volume} = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\Delta x \stackrel{\text{def}}{=} \int_a^b A(x)\, dx,$$

where $A(x)$ gives the area of the cross-section perpendicular to the x-axis at every x between $x = a$ and $x = b$.

Sometimes slicing horizontal to the y-axis may be more convenient:

Question 1:

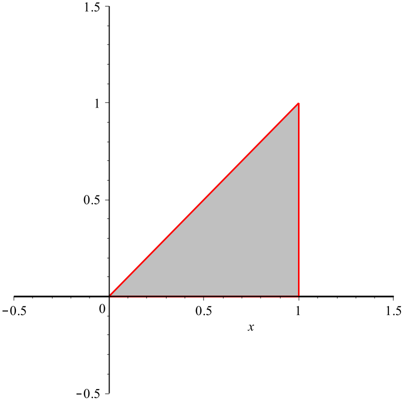
Let R be the rectangle formed by the x-axis, the y-axis, and the lines $y = 1$ and $x = 3$.



What shape is the solid formed when R is rotated about the x-axis?

Question 2:

Let T be the triangle formed by the lines $y = x$, $x = 1$ and the x-axis.



What shape is the solid formed when T is rotated about the x-axis?

Finding Volume with Slices

To find the volume of an object:

- We have seen that if the area of the cross-sections perpendicular to a horizontal axis of revolution at the point x are represented by $A(x)$ for all $x \in [a, b]$, then then

$$\text{Volume} = \int_a^b A(x)\, dx$$

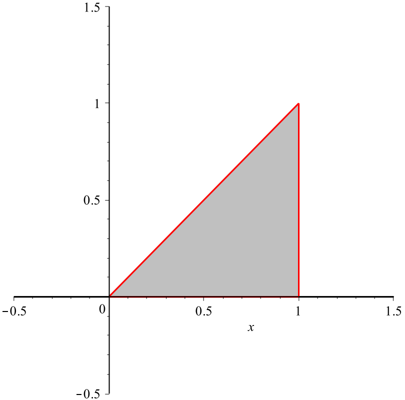
- Similarly, if the area of the cross-sections perpendicular to a vertical axis of revolution at the point y are represented by $A(y)$ for all $y \in [c, d]$, then

$$\text{Volume} = \int_c^d A(y)\, dy$$

- To find volume, we thus need to first figure out what $A(x)$ or $A(y)$ is.

Revisiting Question 2:

Let T be the triangle formed by the lines $y = x$, $x = 1$ and the x-axis.



Use this technique to find the volume of the solid formed when T is rotated about the x-axis.

In Class Work

For each three dimensional object described below,

(a) Sketch the object

(b) Set up an integral that gives you the volume of the object

(c) Evaluate the integral to find the volume

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1. The solid formed when the region bounded by $y = x^2$, $x = -2$, $x = 2$, and $y = 0$ is rotated about the x-axis.
2. The solid formed when the region bounded by $y = x^2$, $x = 0$, and $y = 4$ is rotated about the y-axis.

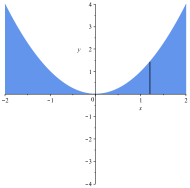
   Do the circular cross-sections go from a smallest x to a largest x? Is the radius a function of x? Is the infinitesimal thickness dx (like Δx)?

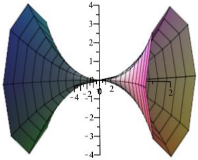
3. The solid formed when the region bounded by $x = y^{1/3}$, $x = -1$, and $y = 8$ is rotated about the line $x = -1$.

   Circular cross-sections have area πR^2. What is R?

Solutions

1. The solid formed when the region bounded by $y = x^2$, $x = -2$, $x = 2$, and $y = 0$ is rotated about the x-axis.





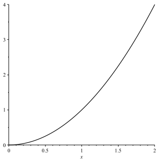
- Cross-sections have radius $r = f(x) = x^2$ for all $x \in [-2, 2]$
- Thus $A(x) = \pi \left[x^2\right]^2$, so

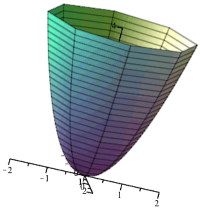
$$V = \int_{-2}^{2} \pi x^4 \, dx = \frac{\pi}{5} x^5 \Big|_{-2}^{2} = \frac{\pi}{5}\left((2)^5 - (-2)^5\right) = \frac{64\pi}{5}$$

Solutions

2. The solid formed when the region bounded by $y = x^2$, $x = 0$, and $y = 4$ is rotated about the y-axis.

   The solid is formed *inside* the rotated parabola. This solid is almost completely unrelated to the one found in the previous problem.



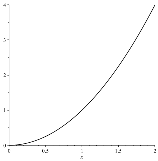


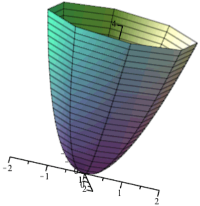
   In this situation,

   - the cross-sections are going from $y = 0$ to $y = 4$.
   - the radius depends on how high up the y-axis we've moved, that is $r = f(y)$. Solving $y = x^2$ for x, we find $r(y) = x = \sqrt{y}$
   - the thickness of the cross-sections is measured along the y-axis, so dy

Solutions

2. (continued) The solid formed when $x = \sqrt{y}$, y in $[0, 4]$, is rotated about the y-axis.





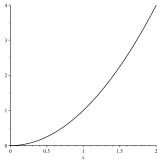
In this situation,

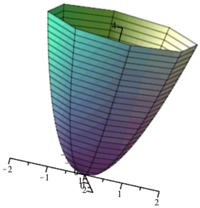
- the cross-sections are going from $y = 0$ to $y = 4$.
- the radius depends on how high up the y-axis we've moved. $r = f(y) = \sqrt{y}$
- the thickness of the cross-sections is measured along the y-axis, so dy

$$\text{Volume} = \int_0^4 \pi \left[f(y)\right]^2 \, dy = \int_0^4 \pi \left(\sqrt{y}\right)^2 \, dy = \int_0^4 \pi y \, dy$$

Solutions

2. (continued) The solid formed when $x = \sqrt{y}$, y in $[0, 4]$, is rotated about the y-axis.





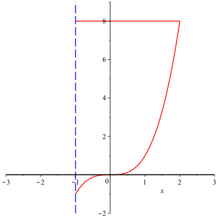
$$\text{Volume} = \int_0^4 \pi \left[f(y)\right]^2 \, dy = \int_0^4 \pi \left(\sqrt{y}\right)^2 \, dy = \int_0^4 \pi y \, dy$$

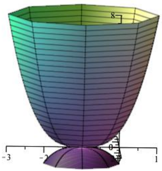
And so,

$$\text{Volume} = \frac{\pi}{2} y^2 \Big|_0^4 = \frac{\pi}{2}(16 - 0) = 8\pi$$

Solutions

3. The solid formed when the region bounded by $x = y^{1/3}$, $x = -1$, and $y = 8$ is rotated about the line $x = -1$.

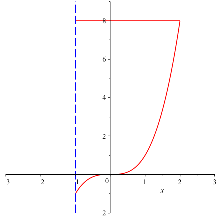


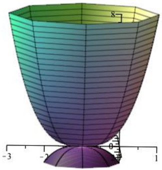


- As in #2, these washers are perp to y-axis so radius is a fn of y and the thickness is dy.
- The function $x = y^{1/3}$ only gives the distance from the function to the y-axis. The radius goes from the function to the line $x = -1$.
  $R(y) = f(y) - (-1) = y^{1/3} + 1$
- Thus $V = \int_{y=-1}^{y=8} A(y)\,dy = \int_{-1}^{8} \pi\left[\left(y^{1/3}+1\right)^2\right]\,dy$

Solutions

3. (continued) The solid formed when the region bounded by $x = y^{1/3}$, $x = -1$, and $y = 8$ is rotated about the line $x = -1$.





$$
\begin{aligned}
V &= \pi \int_{-1}^{8} \left[\left(y^{1/3} + 1 \right)^2 \right]^2 \, dy \\
&= \pi \int_{-1}^{8} y^{2/3} + 2y^{1/3} + 1 \, dy \\
&= \pi \left[\frac{3}{5} y^{5/3} + \frac{3}{2} y^{4/3} + y \right]_{-1}^{8} = \cdots = \frac{513\pi}{10}
\end{aligned}
$$