Solid of Revolution

If we have a **solid of revolution** ...

- ... formed by rotating a function $f(x)$ about a horizontal line, then

  $$\text{Volume} = \int_a^b \pi \left[R(x)\right]^2 \, dx$$

  where $R(x)$ is the radius from the axis of revolution to the curve for $x \in [a, b]$.
- ... formed by rotating a function $g(y)$ about a vertical line, then

  $$\text{Volume} = \int_c^d \pi \left[R(y)\right]^2 \, dy$$

  where $R(y)$ is the radius from the axis of revolution to the curve for $y \in [c, d]$.

In Class Work

For each three dimensional object described below,

(a) Sketch the object

(b) Set up an integral that gives you the volume of the object

(c) Evaluate the integral to find the volume

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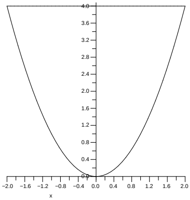
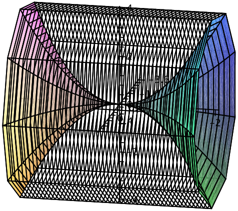
1. The solid formed when the region bounded by $y = x^2$ and $y = 4$ is rotated about the x-axis.
2. The solid formed when the region bounded by $x = \sqrt{y}$, $x = -\sqrt{y}$, and $y = 4$ (that is, the same region as in Problem 1) is rotated about the line $x = 3$.

Solutions

1. The solid formed when the region bounded by $y = x^2$ and $y = 4$ is rotated about the x-axis.

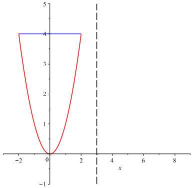




$$\begin{aligned}
\text{Volume} &= 2 \cdot \int_0^2 A(x)\, dx = 2\pi \int_0^2 (\text{outer } r)^2 - (\text{inner r})^2\, dx \\
&= 2\pi \int_0^2 (4)^2 - (x^2)^2\, dx = 2\pi \int_0^2 16 - x^4\, dx \\
&= 2\pi (16x - \frac{x^5}{5}) \Big|_0^2 = 2\pi[(32 - \frac{32}{5}) - 0] = 2\pi(\frac{128}{5}) = \frac{256\pi}{5}
\end{aligned}$$

Solutions

2. The solid formed when the region bounded by $x = \sqrt{y}$, $x = -\sqrt{y}$, and $y = 4$ (that is, the same region as in Problem 1) is rotated about the line $x = 3$.

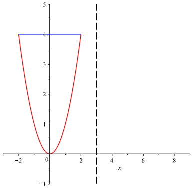


Notice: While this is the same region we rotated in #1, because we're rotating around $x = 3$, the solid formed will have a hole.

- As in #1, the cross-sections of interest are washers rather than disks
- The cross-sections are perpendicular to the y-axis rather than to the x-axis, so we'll be integrating with respect to y.
- Thus $V = \int_{y=0}^{y=4} A(y)\, dy = \int_0^4 \pi\left(\left(R_{\text{outer}}\right)^2 - \left(R_{\text{inner}}\right)^2\right) dy$
- Our axis of rotation is $x = 3$.
  - $R_{\text{outer}} =$ distance from $x = 3$ to $x = -\sqrt{y} = 3 + \sqrt{y}$
  - $R_{\text{inner}} =$ distance from $x = 3$ to $x = \sqrt{y} = 3 - \sqrt{y}$

Solutions

2. (continued) The solid formed when the region bounded by $x = \sqrt{y}$, $x = -\sqrt{y}$, and $y = 4$ (that is, the same region as in Problem 1) is rotated about the line $x = 3$.



$$V = \int_0^4 \pi\left(\left(R_{\text{outer}}\right)^2 - \left(R_{\text{inner}}\right)^2\right)\, dy$$

where $R_{\text{outer}} = 3 + \sqrt{y}$ and $R_{\text{inner}} = 3 - \sqrt{y}$

$$\begin{aligned} V &= \pi \int_0^4 \left(3+\sqrt{y}\right)^2 - \left(3-\sqrt{y}\right)^2\, dy \\ &= \pi \int_0^4 \left(9 + 6\sqrt{y} + y\right) - \left(9 - 6\sqrt{y} + y\right)\, dy \\ &= \pi \int_0^4 12y^{1/2}\, dy = 12\pi\left(\frac{2}{3}y^{3/2}\right)\Big|_0^4 = 64\pi \end{aligned}$$