Question:

Suppose $f(x)$ is a differentiable function on $[a, b]$. What is the arc length of the graph $y = f(x)$ on $[a, b]$?

It turns out the answer is

$$\text{Arclength} = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

$\vdots$

But why??

Recall - Mean Value Theorem

Suppose that g is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Then there is a number c between a and b for which

$$g'(c) = \frac{g(b) - g(a)}{b - a}.$$

In Class Work

Find the arc length (exactly, when possible; otherwise, approximate using the midpoint method within 0.001).

1. $y = 4x^{3/2} + 1$ over the interval $[1, 2]$
2. $y = x^3 - 1$ over the interval $[-1, 1]$

Solutions

1. $y = 4x^{3/2} + 1$ over the interval $[1, 2]$

$$\begin{aligned}\text{Arclength} &= \int_1^2 \sqrt{1 + \left[f'(x)\right]^2}\, dx = \int_1^2 \sqrt{1 + \left(4 \cdot \frac{3}{2}x^{1/2} + 0\right)^2}\, dx \\ &= \int_1^2 \sqrt{1 + \left(6x^{1/2}\right)^2}\, dx = \int_1^2 \sqrt{1 + 36x}\, dx\end{aligned}$$

Let $u = 1 + 36x$. Then $du = 36\, dx$, or $\frac{1}{36}\, du = dx$.

$$\begin{aligned}\text{Arclength} &= \frac{1}{36}\int_{x=1}^{x=2} u^{1/2}\, du = \frac{1}{36} \cdot \frac{2}{3}(1 + 36x)^{3/2}\Big|_1^2 \\ &= \frac{1}{54}\left[(1 + 72)^{3/2} - (1 + 36)^{3/2}\right] = \frac{73\sqrt{73} - 37\sqrt{37}}{54}\end{aligned}$$

Solutions

2. $y = x^3 - 1$ over the interval $[-1, 1]$

$$\begin{aligned} \text{Arclength} &= \int_{-1}^{1} \sqrt{1 + \left[f'(x)\right]^2}\, dx = \int_{-1}^{1} \sqrt{1 + \left(3x^2 - 0\right)^2}\, dx \\ &= \int_{-1}^{1} \sqrt{1 + 9x^4}\, dx \quad \text{Can't antidifferentiate.} \end{aligned}$$

Use the midpoint method and technology to approximate within 0.001. We need to find the number of subintervals to use.

Let $g(x) = \sqrt{1 + 9x^4}$.

Recall: $|\mathcal{I} - M_n| \leq \dfrac{M\big(1 - (-1)\big)^3}{24n^2} = \dfrac{M}{3n^2}$, where $M \geq |g''(x)|$.

Solutions

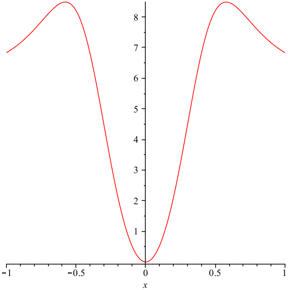
2. (continued)

For $y = x^3 - 1$ over the interval $[-1, 1]$, Arclength$= \int_{-1}^{1} \sqrt{1 + 9x^4}\, dx$

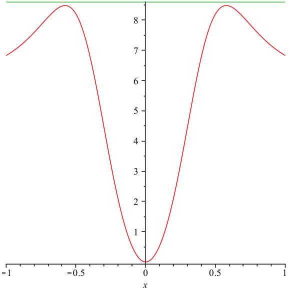
$|\mathcal{I} - M_n| \leq \frac{M}{3n^2}$, where $M \geq |g''(x)|$, using $g(x) = \sqrt{1 + 9x^4}$.

Finding the number of subintervals to use in M_n to be within 0.001:

Using technology to find and graph $|g''(x)|$, we find that



Graph of $|g''(x)|$



Graph of $|g''(x)|$ and $y = 8.6$

Can use $M = 8.6$

Solutions

2. (continued)

For $y = x^3 - 1$ over the interval $[-1, 1]$, Arclength$= \int_{-1}^{1} \sqrt{1 + 9x^4}\, dx$

$|\mathcal{I} - M_n| \leq \frac{M}{3n^2}$, where $M \geq |g''(x)|$, using $g(x) = \sqrt{1 + 9x^4}$.

Finding the number of subintervals to use in M_n to be within 0.001:

Using technology to find and graph $|g''(x)|$, we found that we can use $M = 8.6$

$$|\mathcal{I} - M_n| \leq \frac{8.6}{3n^2}$$

Since we want the error to be less than or equal to 0.001, we need to find n so that

$$\frac{8.6}{3n^2} \leq \frac{1}{1000} \Rightarrow \frac{8600}{3} \leq n^2 \Rightarrow n \geq \sqrt{\frac{8600}{3}} \approx 53.54$$

Use $n = 54$. Using technology again (Maple!), $M_{54} \approx 3.095080620$, so $\mathcal{I}$ is within 0.001 of 3.095080620