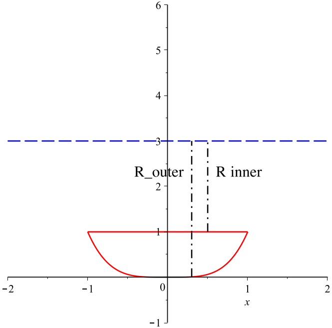
Daily WeBWorK Problem 1

1. Volume - rotate region bed by $y = x^4$ and $y = 1$ about $y = 3$.
   $y = x^4$ and $y = 1$ intersect at $x = \pm 1$.

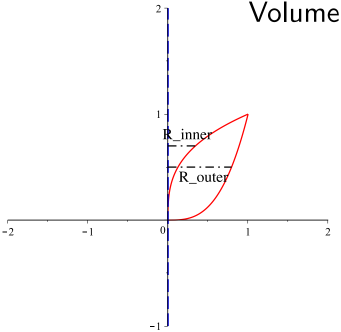


$$
\begin{aligned}
\text{Volume} &= V_{\text{outer solid}} - V_{\text{inner solid}} \\
&= \pi \int_{-1}^{1} (R_{\text{outer}})^2 - (R_{\text{inner}})^2 \, dx \\
&= \pi \int_{-1}^{1} (3 - x^4)^2 - (3-1)^2 \, dx \\
&= \pi \int_{-1}^{1} 9 - 6x^4 + x^8 - 4 \, dx \\
&= \pi \left(5x - \frac{6}{5}x^5 + \frac{1}{9}x^9 \right) \Bigg|_{-1}^{1} \\
&= \pi \left[\left(5 - \frac{6}{5} + \frac{1}{9} \right) - \left(-5 + \frac{6}{5} - \frac{1}{9} \right)
\end{aligned}
$$

Daily WeBWorK Problem 2

1. Volume - rotate region bed by $x = y^3$ and $x = y^{1/3}$ about the y-axis.

$x = y^3$ and $x = y^{1/3}$ intersect at $y = 0$ and $y = 1$ (also $y = -1$).



$$\begin{aligned}
\text{Volume} &= V_{\text{outer solid}} - V_{\text{inner solid}} \\
&= \pi \int_0^1 (R_{\text{outer}})^2 - (R_{\text{inner}})^2 \, dy \\
&= \pi \int_0^1 (y^{1/3})^2 - (y^3)^2 \, dy \\
&= \pi \int_0^1 y^{2/3} - y^6 \, dy \\
&= \pi \left(\frac{3}{5} y^{5/3} - \frac{1}{7} y^7 \right) \Bigg|_0^1 \\
&= \pi \left[\left(\frac{3}{5} - \frac{1}{7} \right) - \left(0 \right) \right]
\end{aligned}$$

Recall: The Fundamental Theorem of Calculus

If f is continuous on $[a, b]$ and F is any antiderivative of f, then

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

Notice:

- This requires that f is continuous on the entire interval $[a, b]$.
- The choice to write the interval as $[a, b]$ is very deliberate - the interval of integration must be closed and bounded.

Question:

What happens to the area $\int_1^B \frac{1}{x^2}\, dx$ as B gets bigger and bigger?

Improper Integrals over Unbounded Intervals

- If $f(x)$ is continuous on the interval $[a, \infty)$, we define the improper integral $\int_a^\infty f(x)\,dx$ to be

$$\int_a^\infty f(x)\,dx \stackrel{\text{def}}{=} \lim_{B\to\infty} \int_a^B f(x)\,dx.$$

- Similarly, if $f(x)$ is continuous on the interval $(-\infty, b]$, we define

$$\int_{-\infty}^b f(x)\,dx \stackrel{\text{def}}{=} \lim_{A\to-\infty} \int_A^b f(x)\,dx.$$

In either case, if the limit exists (and equals some value L), we say that the improper integral converges (to L). If the limit does not exist (whether because it is infinite or for other reasons), we say that the improper integral diverges.

Questions to Ponder:

- Why did the first improper integral ($\int_1^\infty \frac{1}{x^2}\,dx$) converge to 1, while the other three ($\int_0^\infty 2\,dx$, $\int_0^\infty 10^{-100}\,dx$, and $\int_0^\infty f(x)\,dx$, where $\lim_{x\to\infty} f(x) = 10^{-100}$) instead diverged?

- That is, why did $\frac{1}{x^2}$ have finite area over the unbounded interval $[1,\infty)$?

In Class Work

1. As $x \to \infty$, does each *integrand* diverge or converge (if so, to what?) Also, does each improper *integral* diverge or converge (if so, to what?)

   a. $\displaystyle\int_1^\infty \frac{1}{x^3}\,dx$ b. $\displaystyle\int_1^\infty 1 + \frac{1}{x^2}\,dx$ c. $\displaystyle\int_1^\infty \frac{1}{x}\,dx$ d. $\displaystyle\int_0^\infty xe^{-x^2}\,dx$

2. Think about all the results you've seen, as well as the big picture.
   (a) Is it *necessary* that $f(x)$ converge to 0 as $x \to \infty$ in order for $\int_a^\infty f$ to converge to a finite number?
   (b) If $f(x)$ *does* converge to 0 as $x \to \infty$, *must* $\int_a^\infty f$ converge to a finite number? That is, is $f(x) \to 0$ a *sufficient* condition for $\int_a^\infty f$ to converge to a finite number?

Important Lessons:

1. There is a huge distinction between $f(x)$ converging – that is, $\lim_{x\to\infty} f(x)$ being finite – and $\int_a^\infty f(x)\,dx$ converging. Just because you can find $\lim_{x\to\infty} f(x)$, and it's a finite number, does **not** mean that $\int_a^\infty f(x)\,dx$ will be finite.

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2. In fact, if $\lim_{x\to\infty} f(x)$ exists **but is not 0**, $\int_a^\infty f$ diverges! No need to investigate any further.

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2. In fact, if $\lim_{x\to\infty} f(x)$ exists **but is not 0**, $\int_a^\infty f$ diverges! No need to investigate any further.
3. If $\lim_{x\to\infty} f(x)$ **is** 0, $\int_a^\infty f$ may converge or it may diverge – to find out, you must actually do the antidifferentiation and the limit.

Solutions

1(a) $\displaystyle\int_1^\infty \frac{1}{x^3}\,dx$

- $1/x^3$ converges to 0 as $x \to \infty$
- $$\begin{aligned}\int_1^\infty \frac{1}{x^3}\,dx &= \lim_{R\to\infty}\int_1^R x^{-3}\,dx = \lim_{R\to\infty}\left(-\frac{1}{2}\cdot\frac{1}{x^2}\right)\Big|_1^R \\ &= \lim_{R\to\infty}\left(-\frac{1}{2R^2}+\frac{1}{2}\right) = \frac{1}{2}\end{aligned}$$

  This improper integral converges, to $\frac{1}{2}$.
- **Results:** Integrand converges to 0; integral converges.

Solutions

1(b) $\int_1^\infty 1 + \frac{1}{x^2}\, dx$

- $1 + \frac{1}{x^2} \to 1$ as $x \to \infty$
- $$\int_1^\infty 1 + \frac{1}{x^2}\, dx = \int_1^\infty 1\, dx + \int_1^\infty \frac{1}{x^2}\, dx$$

  We've seen that the first integral on the right diverges (to ∞), the second one converges (to 1).

  Because this sum does not approach a finite number, it *diverges*.
- **Result:** Integrand converges to 1; integral diverges.

Solutions

1(c) $\displaystyle\int_1^\infty \frac{1}{x}\,dx$

- $\frac{1}{x} \to 0$ as $x \to \infty$
-

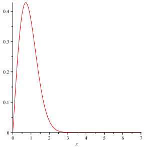
$$\begin{aligned}\int_1^\infty \frac{1}{x}\,dx &= \lim_{R\to\infty}\int_1^R \frac{1}{x}\,dx = \lim_{R\to\infty} \ln(x)\Big|_1^R \\ &= \lim_{R\to\infty}(\ln(R)-\ln(1)) = \lim_{R\to\infty}\ln(R) = \infty\end{aligned}$$

  This improper integral diverges (slowly).
- **Result:** Integrand converges to 0; integral diverges.

Solutions

1(d) $\displaystyle\int_0^{\infty} xe^{-x^2}\,dx$

As $x \to \infty$, $\frac{x}{e^{x^2}} \to \frac{\infty}{\infty}$. Another limit we can't do b/c it's in *indeterminate form*! Looking at a graph of xe^{-x^2}, can see that integrand approaches 0.



- Let $u = -x^2$, so $du = -2x\,dx$, or $-\frac{1}{2}\,du = x\,dx$
  Also, $x = 0 \Rightarrow u = 0$; $x = \infty \Rightarrow u = -\infty$.

$$\int_0^{\infty} xe^{-x^2}\,dx = -\frac{1}{2}\int_0^{-\infty} e^u\,du = \frac{1}{2}\int_{-\infty}^{0} e^u\,du = \frac{1}{2}\lim_{R\to\infty} e^u\Big|_{-R}^{0}$$

$$= \frac{1}{2}\lim_{R\to\infty} 1 - e^{-R} = \frac{1}{2}\left(1 - \lim_{R\to\infty}\frac{1}{e^R}\right) = \frac{1}{2}$$

The improper integral converges, to 1.

- **Result:** Integrand converges to 0; integral converges.

Solutions

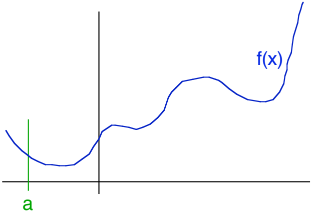
2. Think about all the results you've seen, as well as the big picture.
   (a) Is it *necessary* that $f(x)$ converge to 0 as $x \to \infty$ in order for $\int_a^\infty f(x)\,dx$ to converge to a finite number?

Convergent integral	What f converges to
$\int_1^\infty \frac{1}{x^2}\,dx$	$\frac{1}{x^2} \to 0$
$\int_1^\infty \frac{1}{x^3}\,dx$	$\frac{1}{x^3} \to 0$
$\int_0^\infty xe^{-x^2}\,dx$	$xe^{-x^2} \to 0$

For what it's worth, so far every example that we've seen of a convergent improper integral *has* had an integrand that converges to 0 as $x \to \infty$. But that's not enough.

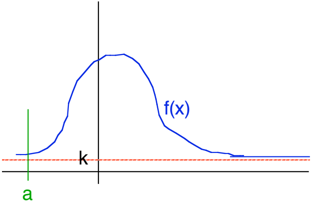
Solutions:

2(a) (continued)



As $x \to \infty$, $f(x) \to \infty$,

$$\text{and } \int_a^\infty f(x)\,dx = \infty$$

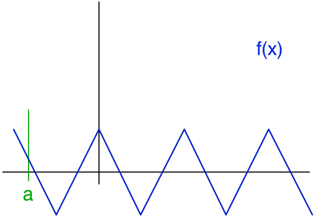


As $x \to \infty$, $f(x) \to k \neq 0$, and

$$\begin{aligned}\int_a^\infty f(x)\,dx &= \lim_{R\to\infty}\int_a^R f(x)\,dx \\ &> \lim_{R\to\infty}\int_a^R k\,dx \\ &> \lim_{R\to\infty} kR = \pm\infty\end{aligned}$$

Solutions:

2(a) (continued)



As $x \to \infty$, $\lim_{x\to\infty} f(x)$ d.n.e., and $\int_a^\infty f(x)\,dx = \lim_{R\to\infty} f(x)\,dx$ d.ne., so the integral diverges

Conclusion: The only way $\int_a^\infty f(x)\,dx$ can have a **hope** of converging to a finite number is if $\lim_{x\to\infty} f(x) = 0$.

In other words, if $\lim_{x\to\infty} f(x) \neq 0$, $\int_a^\infty f(x)\,dx$ **must** diverge.

Solutions:

2(b) If $f(x)$ does converge to 0 as $x \to \infty$, *must* $\int_a^b f(x)\,dx$ automatically converge to a finite number? That is, is $f(x) \to 0$ a *sufficient* condition for $\int_a^\infty f(x)\,dx$ to converge to a finite number?

Functions that converge to 0	What $\int_a^\infty f(x)\,dx$ does
xe^{-x^2}	converges
$\frac{1}{x^2}$	converges
$\frac{1}{x^3}$	converges
$\frac{1}{x}$	diverges

Thus knowing that $f(x) \to \infty$ is *not* sufficient information to conclude that $\displaystyle\int_a^\infty f(x)\,dx$ converges!