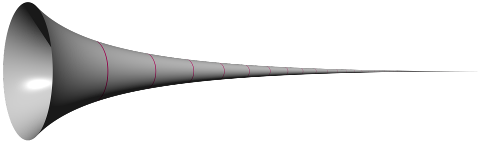
Gabriel's Trumpet

Gabriel with his trumpet

Math-y version of Gabriel's Trumpet



Take the region R btwn $y = \dfrac{1}{x}$ and $y = 0$ on $[1, \infty)$.
Form Gabriel's Trumpet by rotating the region R about the x-axis.

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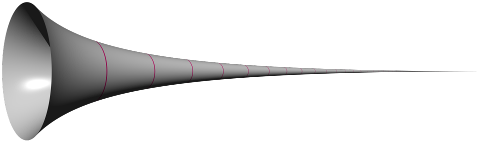

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- Volume? $V = \pi \displaystyle\int_1^\infty \frac{1}{x^2}\,dx = \pi$

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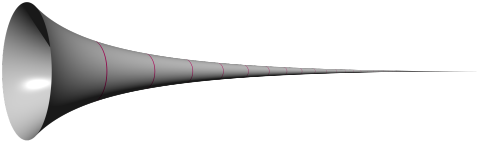

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- Area of R? $A = \int_1^\infty \frac{1}{x}\, dx = \infty$

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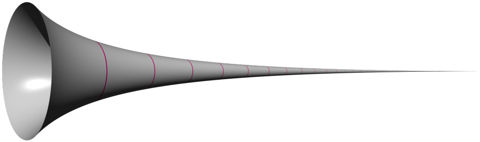

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- Surface area?

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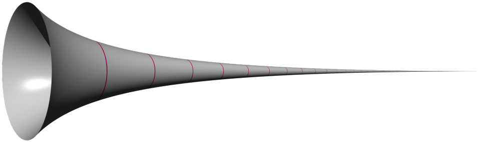

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Form Gabriel's Trumpet by rotating the region R about the x-axis.

- Volume? $V = \pi \displaystyle\int_1^\infty \frac{1}{x^2}\, dx = \pi$
- Area of R? $A = \displaystyle\int_1^\infty \frac{1}{x}\, dx = \infty$
- Surface area? $SA = 2\pi \displaystyle\int_1^\infty \frac{1}{x}\sqrt{1 + \frac{1}{x^4}}\, dx = \ldots = \infty$

Definition:

If f is continuous on the interval $[a, b)$, but $|f(x)| \to \infty$ as $x \to b^-$, we define the **improper integral** of f on $[a, b]$ by

$$\int_a^b f(x)\,dx = \lim_{B \to b^-} \int_a^B f(x)\,dx.$$

Similarly, if f is continuous on the interval $(a, b]$, but $|f(x)| \to \infty$ as $x \to a^+$, we define the improper integral

$$\int_A^b f(x)\,dx = \lim_{A \to a^+} \int_A^b f(x)\,dx.$$

In either case, if the limit exists and equals some finite value L, we say that the improper integral **converges** to L. If the limit does not exist (whether because it equals $\pm\infty$ or because for some other reason), we say that the improper integral **diverges**.

Important Point:

If $\lim_{x\to c} f(x) = \pm\infty$ and if $a \le c \le b$, $\int_a^b f(x)\,dx$ may or may not be infinite.

Examples:

Both $\frac{1}{\sqrt{x}}$ and $\frac{1}{x^2} \to \infty$ as $x \to 0$.

BUT $\int_0^1 \frac{1}{\sqrt{x}}\,dx = 2$, while $\int_0^1 \frac{1}{x^2}\,dx = \infty$.

Question:

$\displaystyle\int_1^\infty \frac{1}{x^n}\,dx$ behaves differently depending on what n is.

$$\int_1^\infty \frac{1}{x^n}\,dx \qquad \begin{cases} \textit{diverges} & \textit{for any } n \le 1 \\ \textit{converges to } \dfrac{1}{n-1} & \textit{for any } n > 1 \end{cases}$$

Notice that as long as $n > 0$*, the* integrand *will always converge to 0, but that the* integral *only converges (not to 0) when* $n > 1$*.*

Also notice: All of the above is true for $\displaystyle\int_a^\infty \frac{1}{x^n}\,dx$ *for any* $a > 0$*, not just for* $a = 1$*.*

Question: Does $\displaystyle\int_0^1 \frac{1}{x^n}\,dx$ behave differently depending on what n is?

Conclusion:

$$\int_0^1 \frac{1}{x^n}\,dx \qquad \begin{cases} \text{diverges} & \text{for any } n \geq 1 \\ \text{converges to } \dfrac{1}{n-1} & \text{for any } n < 1 \end{cases}$$

Notice that as long as $n > 0$, the *integrand* will always diverge to ∞ as $x \to 0^+$, but that the *integral* only diverges (to ∞) when $n > 1$.

Also notice: All of the above is true for $\displaystyle\int_0^b \frac{1}{x^n}\,dx$ for any $b > 0$, not just for $b = 1$.

Question:

Both $\int_2^\infty \frac{2}{x^4+5}\,dx$ and $\int_0^4 \frac{1}{x^{1/2}+3x^{1/3}}\,dx$ are improper – the first region is unbounded on the right, the second is unbounded above.

- Do they converge?
- If one (or both) converges, what does it converge to?

Neither of these is easy to antidifferentiate. But we *could* use approximation techniques (we'd have to bound the interval, or omit the problem spot) ... *if* we already know the integrals converge.

(Otherwise, it would be a huge waste of time.)