Improper Integrals - Facts to make part of you

- $\int_0^b \frac{1}{x^p}\,dx$ converges (to $\frac{1}{1-p}b^{1-p}$) if $0 < p < 1$, diverges for $p \geq 1$.
- $\int_a^\infty \frac{1}{x^p}\,dx$ converges (to $\frac{1}{(p-1)a^{p-1}}$) if $p > 1$, diverges for $p \leq 1$.
- If $\lim_{x\to\infty} f(x) \neq 0$, $\int_a^\infty f(x)\,dx$ **must** diverge.
- If $\lim_{x\to\infty} f(x) = 0$, $\int_a^\infty f(x)\,dx$ *may* converge (like $\int_a^\infty \frac{1}{x^p}\,dx$ with $p > 1$), or it *may* diverge (like $\int_a^\infty \frac{1}{x^p}\,dx$ with $p \leq 1$).

  In other words – $\lim_{x\to\infty} f(x) = 0$ **tells you NOTHING about the convergence or divergence of** $\int_a^\infty f(x)\,dx$.

Question:

Both $\int_2^{\infty} \frac{2}{x^4+5}\,dx$ and $\int_0^4 \frac{1}{x^{1/2}+3x^{1/3}}\,dx$ are improper – the first region is unbounded on the right, the second is unbounded above.

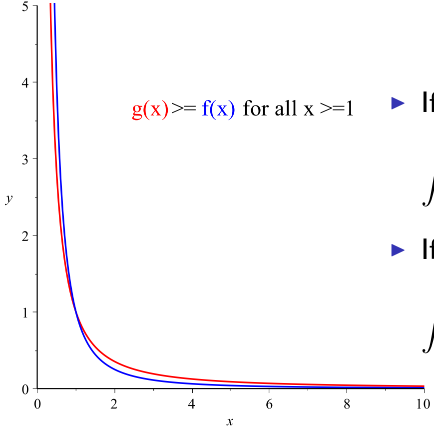
- Do they converge?
- If one (or both) converges, what does it converge to?

Neither of these is easy to antidifferentiate. But we *could* use approximation techniques (we'd have to bound the interval, or omit the problem spot) ... *if* we already know the integrals converge.

(Otherwise, it would be a huge waste of time.)

Comparison Test for Improper Integrals

Assume that on $[a, \infty)$, $f(x)$ & $g(x)$ are continuous; $0 \leq f(x) \leq g(x)$



- If $\displaystyle\int_a^{\infty} g(x)\,dx$ converges, then $\displaystyle\int_a^{\infty} f(x)\,dx$ also converges.
- If $\displaystyle\int_a^{\infty} f(x)\,dx$ diverges, then $\displaystyle\int_a^{\infty} g(x)\,dx$ also diverges.

Note: If $\displaystyle\int_a^{\infty} f(x)\,dx$ converges, or if $\displaystyle\int_a^{\infty} g(x)\,dx$ diverges, we don't have enough information to conclude what the other integral does.

When Choosing Comparison Integrals

- You must convince yourself, as well as whoever is reading your work, that the relationship between your integrand and your comparison integrand that you think is true **is** true – you can't just *say* it's true.
  - Use arithmetic or algebra, not graphs.
  - Choosing your comparison function by looking at graphs is not convincing.
  - What you see in your graphing window doesn't mean the two functions are in that same relationship all the way out to infinity.
- It's best to find comparison functions that you can pretty easily compare to the original using simple arithmetic, as in the example we did.
- You must convince yourself, as well as your reader, that your comparison is useful

Examples:

What would be your first choice for comparison integrals? (Note: not all will actually be helpful)

1. $\displaystyle\int_2^\infty \frac{1}{\sqrt{x}-1}$
2. $\displaystyle\int_1^\infty \frac{1}{x^2+x}$
3. $\displaystyle\int_0^\infty \frac{1}{e^x+2}$

4. $\displaystyle\int_1^\infty \frac{1}{x+\cos(x)}$

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Try comparing to $\displaystyle\int_2^\infty \frac{1}{\sqrt{x}}\,dx$

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2. $\displaystyle\int_1^\infty \frac{1}{x^2+x}$ Try comparing to either $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ or to $\displaystyle\int_1^\infty \frac{1}{x}\,dx$
3. $\displaystyle\int_0^\infty \frac{1}{e^x+2}$
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In Class Work

Use the Comparison Test to determine whether each of the following improper integrals converges or diverges. In each case be sure to discuss with your group whether your comparison integral is in fact helpful and why.

1. $\displaystyle\int_2^\infty \frac{1}{x^3+2}\,dx$
2. $\displaystyle\int_5^\infty \frac{1}{\sqrt{x}-2}\,dx$
3. $\displaystyle\int_2^\infty \frac{2}{\sqrt{x}+x^2}\,dx$
4. $\displaystyle\int_0^\infty \frac{2}{\sqrt{x}+x^2}\,dx$

Solutions:

Determine whether each of the following improper integrals converges or diverges.

1. $\displaystyle\int_2^\infty \frac{1}{x^3+2}\,dx$ First choice: compare to $\displaystyle\int_2^\infty \frac{1}{x^3}\,dx$. Useful?

$$\begin{array}{rcl} x^3+2 & > & x^3 \\ 0 < \dfrac{1}{x^3+2} & < & \dfrac{1}{x^3} \\ 0 \leq \displaystyle\int_2^\infty \frac{1}{x^3+2}\,dx & \leq & \displaystyle\int_2^\infty \frac{1}{x^3}\,dx \end{array}$$

$\displaystyle\int_2^\infty \frac{1}{x^3}\,dx = \int_a^\infty \frac{1}{x^p}\,dx$ with $p > 1$, so converges (to $\frac{1}{2}$).

Thus $0 \leq \displaystyle\int_2^\infty \frac{1}{x^3+2}\,dx \leq \frac{1}{2}$, so it too must converge.

Solutions:

2. $\displaystyle\int_5^\infty \frac{1}{\sqrt{x}-2}\,dx$ First choice: compare to $\displaystyle\int_2^\infty \frac{1}{\sqrt{x}}\,dx$. Useful?

$$\begin{aligned} 0 < \sqrt{x}-2 &< \sqrt{x} \text{ on } [5,\infty) \\ \frac{1}{\sqrt{x}-2} &> \frac{1}{\sqrt{x}} > 0 \text{ on } [5,\infty) \\ \int_5^\infty \frac{1}{\sqrt{x}-2}\,dx &\geq \int_5^\infty \frac{1}{\sqrt{x}}\,dx \geq 0 \end{aligned}$$

$\displaystyle\int_5^\infty \frac{1}{\sqrt{x}}\,dx = \int_5^\infty \frac{1}{x^p}\,dx$ with $p \leq 1$, so diverges to infinity.

Since $\displaystyle\int_5^\infty \frac{1}{\sqrt{x}}\,dx$ diverges (to infinity), and $\displaystyle\int_5^\infty \frac{1}{\sqrt{x}-2}\,dx$ is at least as large, it too must diverge.

Solutions:

3. $\displaystyle\int_2^\infty \frac{2}{\sqrt{x}+x^2}\,dx$ Compare to $\displaystyle\int_2^\infty \frac{2}{\sqrt{x}}\,dx$? $\displaystyle\int_2^\infty \frac{2}{x^2}\,dx$? Which?

$$\begin{aligned}\sqrt{x}+x^2 &> \sqrt{x} \text{ and } x^2 \geq 0\\ 0 \leq \frac{2}{\sqrt{x}+x^2} &< \frac{2}{\sqrt{x}} \text{ and } \frac{2}{x^2}\\ 0 \leq \int_2^\infty \frac{2}{\sqrt{x}+x^2}\,dx &\leq \int_2^\infty \frac{2}{\sqrt{x}}\,dx \text{ and } \int_2^\infty \frac{2}{x^2}\,dx\end{aligned}$$

$\displaystyle\int_2^\infty \frac{2}{\sqrt{x}}\,dx$ diverges to infinity, since $p = \dfrac{1}{2} < 1$.

Knowing that $\displaystyle\int_2^\infty \frac{2}{\sqrt{x}+x^2}\,dx \leq \infty$ is **not** helpful!

$\displaystyle\int_2^\infty \frac{2}{x^2}\,dx$ converges, since $p = 2 > 1$.

Thus $0 \leq \displaystyle\int_2^\infty \frac{2}{\sqrt{x}+x^2}\,dx \leq$ a finite number, so it converges.

Solutions:

4. $\displaystyle\int_0^\infty \frac{2}{\sqrt{x}+x^2}\,dx$ Compare to $\displaystyle\int_0^\infty \frac{2}{\sqrt{x}}\,dx$? $\displaystyle\int_0^\infty \frac{2}{x^2}\,dx$?

Improper at both ends–split into 2 integrals.
Can split at any positive x-value - I choose $x=2$.

$$\int_0^\infty \frac{2}{\sqrt{x}+x^2}\,dx = \int_0^2 \frac{2}{\sqrt{x}+x^2}\,dx + \int_2^\infty \frac{2}{\sqrt{x}+x^2}\,dx$$

Just found $\int_2^\infty \frac{2}{\sqrt{x}+x^2}\,dx$ converges, comparing to $\int_2^\infty \frac{2}{x^2}\,dx$.
How about $\int_0^2 \frac{2}{\sqrt{x}+x^2}\,dx$? Using the same inequalities as before,

$$0 \le \int_0^2 \frac{2}{\sqrt{x}+x^2}\,dx \quad \le \quad \int_0^2 \frac{2}{\sqrt{x}}\,dx \text{ and } \int_0^2 \frac{2}{x^2}\,dx$$

Comparison integral on the far right diverges – useless comparison.
Comparison integral on the near right converges – useful comparison.
Putting it all together, we're adding up two finite pieces, and so the whole thing also converges.