Where We're Going

Recall Goal: To be able to antidifferentiate as many functions as possible.

$\int \frac{1}{1+x^2}\,dx$, $\int \frac{1}{\sqrt{1-x^2}}\,dx$, and $\int x\cos(x^2)\,dx$ – all very basic looking antiderivatives. And yet they are each non-trivial, in their own way.

- (Review) Inverse Trig Functions
- (Review) Their derivatives
- Integration by Substitution

In Class Work from Friday

2. Find the following indefinite or definite integrals, and *check your answers by differentiating*

   (a) $\displaystyle\int \frac{1}{\sqrt{1-x}}\,dx$

   (b) $\displaystyle\int \frac{4}{\sqrt{1-4x^2}}\,dx$

   (c) $\displaystyle\int x\sin(\pi x^2)\,dx$

   (d) $\displaystyle\int_1^3 \frac{x}{1+x^2}\,dx$

   (e) $\displaystyle\int \frac{x}{1+x^4}\,dx$

In Class Work from Friday

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   (a) $\displaystyle\int \frac{1}{\sqrt{1-x}}\,dx = \int \frac{1}{\sqrt{1-x}} \cdot 1\,dx$

   (b) $\displaystyle\int \frac{4}{\sqrt{1-4x^2}}\,dx = \int \frac{1}{\sqrt{1-4x^2}} \cdot 4\,dx$

   (c) $\displaystyle\int x\sin(\pi x^2)\,dx = \int \sin(\pi x^2) \cdot x\,dx$

   (d) $\displaystyle\int_1^3 \frac{x}{1+x^2}\,dx = \int_1^3 \frac{1}{1+x^2} \cdot x\,dx$

   (e) $\displaystyle\int \frac{x}{1+x^4}\,dx = \int \frac{1}{1+x^4} \cdot x\,dx$

In Class Work from Friday

2. Find the following indefinite or definite integrals, and *check your answers by differentiating*

   (a) $\displaystyle\int \frac{1}{\sqrt{1-x}}\,dx = \int \frac{1}{\sqrt{1-x}} \cdot 1\,dx = \int \boxed{(1-x)^{-1}} \cdot 1\,dx$

   (b) $\displaystyle\int \frac{4}{\sqrt{1-4x^2}}\,dx = \int \frac{1}{\sqrt{1-4x^2}} \cdot 4\,dx = \int \boxed{\frac{1}{\sqrt{1-(2x)^2}}} \cdot 4\,dx$

   (c) $\displaystyle\int x\sin(\pi x^2)\,dx = \int \sin(\pi x^2) \cdot x\,dx = \int \boxed{\sin(\pi x^2)} \cdot x\,dx$

   (d) $\displaystyle\int_1^3 \frac{x}{1+x^2}\,dx = \int_1^3 \frac{1}{1+x^2} \cdot x\,dx = \int_1^3 \boxed{(1+x^2)^{-1}} \cdot x\,dx$

   (e) $\displaystyle\int \frac{x}{1+x^4}\,dx = \int \frac{1}{1+x^4} \cdot x\,dx = \int \boxed{\frac{1}{1+(x^2)^2}} \cdot x\,dx$

Question from your reading:

Integration by substitution attempts to undo one of the techniques of differentiation – which one?

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The chain rule

Example:

Differentiate $f(x) = \cos(x^3)$

$f(x)$ is a composition.
Let $u = x^3$, $g(u) = \cos(u)$

f(x)=g(u(x))

$$\begin{aligned} \frac{d}{dx}(f(x)) &= \frac{dg}{du}\frac{du}{dx} \\ &= (-\sin(u))(3x^2) \\ &= -3x^2\sin(x^3) \end{aligned}$$

Question from your reading:

Integration by substitution attempts to undo one of the techniques of differentiation – which one?

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Antidiff $h(x) = -3x^2\sin(x^3)$

$h(x)$ is a product; one piece is a composition.
Let $u = x^3$. Then $\frac{du}{dx} = 3x^2$

$$h(x) = -\sin(u)\frac{du}{dx}$$

Treat $-\sin(u)$ as $\frac{dg}{du}$.
$\Rightarrow g(u) = \cos(u)$.

$$H(x) = \cos(x^3)$$

Concept behind substitution

- $$\frac{d}{dx} f\left(u(x)\right) = f'\left(u(x)\right)\frac{du}{dx}, \text{ so } \int f'\left(u(x)\right)u'(x)\, dx = f\left(u(x)\right) + C$$

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- Without all the x's:

$$\frac{d}{dx} f(u) = f'(u)\frac{du}{dx}, \text{ so } \int f'(u)\frac{du}{dx}\, dx = f(u) + C$$

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- $$\frac{d}{dx} f(u(x)) = f'(u(x))\frac{du}{dx}, \text{ so } \int f'(u(x))\,u'(x)\ dx = f(u(x)) + C$$

- Without all the x's:

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- **Notation:** Rewrite $\frac{du}{dx}$ dx as du

$$\frac{d}{dx} f(u) = f'(u)\frac{du}{dx}, \text{ so } \int f'(u)\ du = f(u) + C$$

Integration by Substitution:

1. Find a composition.
2. Let $u =$ the inside function.
3. Find $\frac{du}{dx}$.
4. Find du: $du = \frac{du}{dx} \cdot dx$.

   If $du = \frac{du}{dx}\, dx$ more or less appears as part of the product (give or take a multiplicative constant), then substitution may work.
5. Substitute in du and u where they go. Can not omit du, and du can not be inside any function (or the denominator of any fraction). No x's can remain at the end of the substitution – all must be replaced by equivalent expressions in u (and one du).
6. Antidifferentiate in u: The antiderivative of $\int f'(u)\, du$ is just $f(u)$.
7. Resubstitute: We now must substitute back in for the original $u(x)$.
8. Check your results by differentiating them!

In Class Work

1. Find the following indefinite or definite integrals, and *check your answers by differentiating*

   (a) $\displaystyle\int \frac{1}{\sqrt{1-x}}\,dx$

   (b) $\displaystyle\int \frac{4}{\sqrt{1-4x^2}}\,dx$

   (c) $\displaystyle\int x\sin(\pi x^2)\,dx$

   (d) $\displaystyle\int_1^3 \frac{x}{1+x^2}\,dx$

   (e) $\displaystyle\int \frac{x}{1+x^4}\,dx$

Solutions

1. Find the following derivatives. Don't worry about algebraic simplifications.

   (a) $$\frac{d}{dx}\big(\arcsin(x^2)\big) = \frac{1}{\sqrt{1-(x^2)^2}} \cdot 2x = \frac{2x}{\sqrt{1-x^4}}$$

   (b) $$\frac{d}{dx}\big(e^x \arctan(4x)\big) = e^x\left(\frac{1}{1+(4x)^2} \cdot 4\right) + e^x \arctan(4x)$$
   $$= \frac{4e^x}{1+16x^2} + e^x \arctan(4x)$$

Solutions:

2(a). Find the following indefinite or definite integrals; check answers.

(a) $\displaystyle\int \frac{1}{\sqrt{1-x}}\,dx$

- Substitute:
  - Composition: $\sqrt{1-x}$.
  - Let $u = 1-x$.
  - Differentiating $u \Rightarrow \dfrac{du}{dx} = -1 \Rightarrow du = -1\,dx$
  - Substitute

$$\int \frac{1}{\sqrt{1-x}}\,dx = \int \frac{1}{\sqrt{1-x}} \cdot -1 \cdot -1\,dx = \int \frac{1}{\sqrt{u}} \cdot -1\,du = -\int u^{-1/2}\,du.$$

- Antidifferentiate in u:

$$-\int u^{-1/2}\,du = -\frac{1}{1/2}u^{1/2} + C = -2\sqrt{u} + C.$$

- Resubstitute:

$$\int \frac{1}{\sqrt{1-x}}\,dx = -2\sqrt{1-x} + C.$$

Check: $\dfrac{d}{dx}\Big(-2\sqrt{1-x} + C\Big) = -2 \cdot \dfrac{1}{2}(1-x)^{-1/2} \cdot (-1) + 0$

Solutions:

2(b) $\displaystyle\int \frac{4}{\sqrt{1-4x^2}}\,dx$

- Substitute:
  - Composition: $\sqrt{1-4x^2} = \sqrt{1-(2x)^2}$
  - Let $u = 2x$.
  - Differentiating $u \Rightarrow \dfrac{du}{dx} = 2 \Rightarrow du = 2\,dx$
  - Substitute:

$$\int \frac{4}{\sqrt{1-4x^2}}\,dx = \int \frac{2}{\sqrt{1-(2x)^2}} \cdot 2\,dx$$

$$= \int \frac{2}{\sqrt{1-u^2}} \cdot du = 2\int \frac{1}{\sqrt{1-u^2}}\,du.$$

- Antidifferentiate in u:

$$2\int \frac{1}{\sqrt{1-u^2}}\,du = 2\arcsin(u) + C.$$

- Resubstitute: $\displaystyle\int \frac{4}{\sqrt{1-4x^2}}\,dx = 2\arcsin(2x) + C.$

Check: $\dfrac{d}{dx}\big(2\arcsin(2x) + C\big) = 2 \cdot \dfrac{1}{\sqrt{1-(2x)^2}} \cdot 2 + 0$

Solutions:

2(c) $\int x \sin(\pi x^2)\, dx$

- Substitute:
  - Composition: $\sin(\pi x^2)$
  - Let $u = \pi x^2$.
  - Differentiating $u \Rightarrow \frac{du}{dx} = 2\pi x \Rightarrow du = 2\pi x\, dx$
  - Substitute:

$$\begin{aligned}\int x \sin(\pi x^2)\, dx &= \int \sin(\pi x^2) \cdot \frac{1}{2\pi} \cdot 2\pi x\, dx \\ &= \int \sin(u) \cdot \frac{1}{2\pi}\, du = \frac{1}{2\pi} \int \sin(u)\, du.\end{aligned}$$

- Antidifferentiate in u:

$$\frac{1}{2\pi} \int \sin(u)\, du = \frac{1}{2\pi} \cdot -\cos(u) + C.$$

- Resubstitute: $\int x \sin(\pi x^2)\, dx = -\frac{1}{2\pi} \cos(\pi x^2) + C.$

Check: $\frac{d}{dx}\left(-\frac{1}{2\pi} \cos(\pi x^2) + C\right) = -\frac{1}{2\pi} \cdot -\sin(\pi x^2) \cdot 2\pi x$

Solutions

2(d) $\displaystyle\int_1^3 \frac{x}{1+x^2}\,dx$

- Substitute:
  - Composition: $\dfrac{x}{1+x^2} = x(1+x^2)^{-1}$.
  - Let $u = 1+x^2$
  - Differentiating $u \Rightarrow \dfrac{du}{dx} = 2x \Rightarrow du = 2x\,dx$
  - Substitute:

$$\int_1^3 \frac{x}{1+x^2}\,dx = \int_{x=1}^{x=3} \frac{1}{1+x^2}\cdot x\,dx = \int_{u=2}^{u=10} \frac{1}{1+x^2}\cdot \frac{1}{2}\cdot 2x\,dx$$
$$= \int_2^{10} \frac{1}{u}\cdot\frac{1}{2}\,du = \frac{1}{2}\int_2^{10}\frac{1}{u}\,du.$$

- Antidifferentiate in u:

$$\frac{1}{2}\int_2^{10}\frac{1}{u}\,du = \frac{1}{2}\ln|u|\Big|_{10}^{2} = \frac{1}{2}\left(\ln|10| - \ln|2|\right) = \frac{1}{2}\ln\left(\frac{10}{2}\right) = \ln(5^{1/2})$$

Check:

$$\frac{d}{dx}\left(\frac{1}{2}\ln|1+x^2|\right) = \frac{1}{2}\cdot\frac{1}{1+x^2}\cdot 2x$$

Solutions

2(e) $\displaystyle\int \frac{x}{1+x^4}\,dx$

- Substitute:
  - Composition: $\frac{x}{1+x^4} = x(1+x^4)^{-1} = x(1+(x^2)^2)^{-1}$.
  - Letting $u = x^4$ won't work. In that case, $du = 4x^3\,dx$, and there is no x^3 term present.
  - Instead, let $u = x^2$.
  - Differentiating $u \Rightarrow \dfrac{du}{dx} = 2x \Rightarrow du = 2x\,dx$
  - Replacing x^2 with u and $x\,dx$ with $\frac{1}{2}\,du$:

$$\int \frac{x}{1+x^4}\,dx = \int \frac{1}{1+(x^2)^2}\cdot\frac{1}{2}\cdot 2x\,dx = \int \frac{1}{1+u^2}\cdot\frac{1}{2}\,du = \frac{1}{2}\int \frac{1}{1+u^2}\,du.$$

- Antidifferentiate in u:

$$\frac{1}{2}\int \frac{1}{1+u^2}\,du = \frac{1}{2}\arctan(u) + C.$$

- Resubstitute:

$$\int \frac{x}{1+x^4}\,dx = \frac{1}{2}\arctan(x^2) + C.$$

Remember to check by differentiating!