Goal:

Want to be able to antidifferentiate as many functions as possible!

- Will be seeing a few applications; there are many more specific to various fields like Physics, Chem, Econ
- Can antidifferentiate most sums and differences of basic building-block functions
- But what about products?

Recall:

There can be no generalized product rule for integration

$\int xe^{x^2}\ dx$

We know:

$$\frac{de^{x^2}}{dx} = 2xe^{x^2},$$

from the chain rule.

So

$$\int xe^{x^2}\ dx = \frac{1}{2}e^{x^2} + C$$

(Use substitution, with $u = x^2$)

$\int xe^x\ dx$

We know:

$$\frac{dxe^x - e^x}{dx} = (xe^x + e^x) - e^x = xe^x,$$

from the product rule.

So

$$\int xe^x\ dx = xe^x - e^x + C$$

But how would we figure that out, if we didn't happen to notice?

Question from your reading:

Integration by Parts attempts to undo one of the techniques of differentiation. Which one is it?

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The Product Rule

Reminder:

Integration by parts:

- If you have an integral of a product, think of it as $\int u\, dv$. Let one factor be u and the other factor, along with dx, be dv.
- Then
$$\int u\, dv = uv - \int v\, du$$
- Must choose dv to be something you can integrate. Most likely to work if v is not more complex than dv was.
- Integration by parts is most likely to work if you can also choose u so that du is less complex than u is.

In Class Work

Evaluate the following integrals using integration by parts, and *check your answers!!*

1. $\int x \cos(x)\ dx$
2. $\int x \ln(x)\ dx$
3. $\int_0^1 x \sin(\pi x)\ dx$
4. $\int x^2 \cos(2x)\ dx$
5. $\int x \sec^2(4x)\ dx$
6. $\int \ln(x)\ dx$

Solutions

1. $\displaystyle\int x\cos(x)\,dx$

$u = x \qquad\qquad dv = \cos(x)\,dx$

$du = dx \qquad\qquad v = \sin(x)$

$$\begin{aligned}\int x\cos(x)\,dx &= uv - \int v\,du \\ &= x\sin(x) - \int \sin(x)\,dx \\ &= x\sin(x) + \cos(x) + C\end{aligned}$$

Verify:

$$\frac{d}{dx}\Big(x\sin(x) + \cos(x) + C\Big) = x\cos(x) + \sin(x) + \big(-\sin(x)\big) = x\cos(x)$$

Solutions

2. $\displaystyle\int x\ln(x)\,dx$

$$u = \ln(x) \qquad dv = x\,dx$$

$$du = \frac{1}{x}\,dx \qquad v = \frac{x^2}{2}$$

$$\begin{aligned}\int x\ln(x)\,dx &= uv - \int v\,du \\ &= \frac{x^2}{2}\ln(x) - \int \frac{x^2}{2}\cdot\frac{1}{x}\,dx = \frac{x^2\ln(x)}{2} - \frac{1}{2}\int x\,dx \\ &= \frac{x^2\ln(x)}{2} - \frac{x^2}{4} + C\end{aligned}$$

Verify:

$$\begin{aligned}\frac{d}{dx}\left(\frac{x^2\ln(x)}{2} - \frac{x^2}{4} + C\right) &= \frac{x^2}{2}\cdot\frac{1}{x} + \frac{\ln(x)}{2}\cdot 2x - \frac{2x}{4} \\ &= \frac{x}{2} + x\ln(x) - \frac{x}{2} = x\ln(x)\end{aligned}$$

Solutions

3. $\displaystyle\int_0^1 x\sin(\pi x)\;dx$

$u = x$ $\qquad$ $dv = \sin(\pi x)\;dx$

$du = dx$ $\qquad$ $v = -\frac{1}{\pi}\cos(\pi x)$

$$\begin{aligned}
\int_0^1 x\sin(\pi x)\;dx &= uv - \int v\;du \\
&= \left(-\frac{x}{\pi}\cos(\pi x) + \frac{1}{\pi}\int \cos(\pi x)\;dx\right)\Bigg|_0^1 \\
&= \left(-\frac{x}{\pi}\cos(\pi x) + \frac{1}{\pi^2}\sin(\pi x)\right)\Bigg|_0^1 \\
&= \left(-\frac{1}{\pi}\cos(\pi) + \frac{1}{\pi^2}\sin(\pi)\right) - \left(0 + \frac{1}{\pi^2}\sin(0)\right) \\
&= -\frac{1}{\pi}\cdot -1 + 0 = \frac{1}{\pi}
\end{aligned}$$

Solutions

4. $\int x^2 \cos(2x)\, dx$

$u = x^2 \qquad dv = \cos(2x)\, dx$

$du = 2x\, dx \qquad v = \frac{1}{2}\sin(2x)$

$$\int x^2 \cos(2x)\, dx = uv - \int v\, du = \frac{x^2}{2}\sin(2x) - \boxed{\int x \sin(2x)\, dx}$$

$u = x \qquad dv = \sin(2x)\, dx$

$du = dx \qquad v = -\frac{1}{2}\cos(2x)$

$$\int x^2 \cos(2x)\, dx = \frac{x^2}{2}\sin(2x) - \boxed{\left[uv - \int v\, du\right]}$$

$$= \frac{x^2}{2}\sin(2x) - \boxed{\left[-\frac{x}{2}\cos(2x) + \frac{1}{2}\int \cos(2x)\, dx\right]}$$

$$= \frac{x^2}{2}\sin(2x) + \boxed{\frac{x}{2}\cos(2x) - \frac{1}{4}\sin(2x) + C}$$

Solutions

5. $\displaystyle\int x \sec^2(4x)\ dx$

$u = x$ $\qquad dv = \sec^2(4x)\ dx$

$du = dx$ $\qquad v = \frac{1}{4}\tan(4x)$

$$\begin{aligned}
\int x \sec^2(4x)\ dx &= uv - \int v\ du \\
&= \frac{1}{4}\left(x\tan(4x) - \int \tan(4x)\ dx\right) \\
&= \frac{1}{4}\left(x\tan(4x) - \int \frac{\sin(4x)}{\cos(4x)}\ dx\right) \\
& \quad \text{Let } u = \cos(4x). \text{Then } -\frac{1}{4}\ du = \sin(4x)\ dx \\
\int x \sec^2(4x)\ dx &= \frac{1}{4}\left(x\tan(4x) + \frac{1}{4}\int \frac{1}{u}\ du\right) \\
&= \frac{x}{4}\tan(4x) + \frac{1}{16}\ln|\cos(4x)| + C
\end{aligned}$$

Solutions

6. $\displaystyle\int \ln(x)\,dx$

$u = \ln(x)$ $\qquad\qquad dv = dx$

$du = \frac{1}{x}\,dx$ $\qquad\qquad v = x$

$$\begin{aligned}\int \ln(x)\,dx &= uv - \int v\,du \\ &= x\ln(x) - \int x \cdot \frac{1}{x}\,dx \\ &= x\ln(x) - \int 1\,dx \\ &= x\ln(x) - x + C\end{aligned}$$

Verify:

$$\begin{aligned}\frac{d}{dx}(x\ln(x) - x + C) &= x \cdot \frac{1}{x} + \ln(x) - 1 \\ &= \ln(x)\end{aligned}$$