WeBWorK, Problems 2 and 3

2. Evaluate $\int \frac{7\,dx}{x\ln(6x)}$

   This can be done using integration by parts or substitution. (Most can not). However, it is much more easily done using substitution. This can be written many ways as a composition; the one where one term is a composition and the other term is (give or take a multiplicative constant) the derivative of the inside of the composition in the first is to write it as

$$7\int \frac{1}{x}\frac{1}{\ln(6x)}\,dx$$

   Let $u = \ln(6x)$, so $du = \frac{1}{6x}\cdot 6\,dx = \frac{1}{x}\,dx$.

3. Evaluate $\int x\sec^2(7x)\,dx$

   See example 4 in Section 5.2; use a mini-substitution for $7x$.

Fun Fact Friday!

Substitution vs Parts

Recall:

- Substitution undoes the chain rule. Try using it when you have a product of two terms, one of which is (or can be thought of as) a composition, and the other of which is (more or less) the derivative of the inside of the composition.
- Parts undoes the product rule. Try using it when you have a product of two unrelated terms.
- Lots of integrals can not be integrated using either method.

Question: Would you use Substitution or Parts on:

- $\int xe^x \, dx$
- $\int xe^{x^2} \, dx$

In Class Work

Evaluate the following integrals using integration by parts, and *check your answers!!*

1. $\displaystyle\int \ln(x)\,dx$
2. $\displaystyle\int e^x \cos(x)\,dx$
3. $\displaystyle\int \sin(\ln(x))\,dx$
4. $\displaystyle\int x^3 e^{x^2}\,dx$

Evaluate the following integrals using substitution or parts, and check your answers.

1. $\displaystyle\int_1^{e^3} \frac{\ln(x)}{x}\,dx$
2. $\displaystyle\int \sin^2(x)\,dx$
3. $\displaystyle\int x^3 \cos(x^4)\,dx$
4. $\displaystyle\int \cos(x^{1/3})\,dx$
5. $\displaystyle\int \arctan(x)\,dx$

Solutions

I(1) $\int \ln(x)\ dx$

$u = \ln(x)$ $\qquad$ $dv = dx$

$du = \frac{1}{x}\ dx$ $\qquad$ $v = x$

$$\begin{aligned}\int \ln(x)\ dx &= uv - \int v\ du \\ &= x\ln(x) - \int x \cdot \frac{1}{x}\ dx \\ &= x\ln(x) - \int 1\ dx \\ &= x\ln(x) - x + C\end{aligned}$$

Verify:

$$\begin{aligned}\frac{d}{dx}(x\ln(x) - x + C) &= x \cdot \frac{1}{x} + \ln(x) - 1 \\ &= \ln(x)\end{aligned}$$

Solutions

I(2) $\int e^x \cos(x)\ dx$

$u = e^x \qquad\qquad dv = \cos(x)\ dx$

$du = e^x\ dx \qquad\qquad v = \sin(x)$

$$\Rightarrow \int e^x \cos(x)\ dx = e^x \sin(x) - \boxed{\int e^x \sin(x)\ dx}$$

$u = e^x \qquad\qquad dv = \sin(x)\ dx$

$du = e^x\ dx \qquad\qquad v = -\cos(x)$

$$\int e^x \cos(x)\ dx = e^x \sin(x) - \boxed{\left(e^x \cdot (-\cos(x)) + \int \cos(x) e^x\ dx\right)}$$

$$2 \int e^x \cos(x)\ dx = e^x \sin(x) + e^x \cos(x)$$

$$\int e^x \cos(x)\ dx = \frac{e^x}{2}\left(\sin(x) + \cos(x)\right) + C$$

Solutions

I(3) $\int \sin(\ln(x))\,dx$

$$u = \sin(\ln(x)) \qquad dv = \ dx$$
$$du = \cos(\ln(x)) \cdot \tfrac{1}{x}\ dx \qquad v = x$$

$$\Rightarrow \int \sin(\ln(x))\,dx = x\sin(\ln(x)) - \boxed{\int (\cos(\ln(x))\,dx}$$

$$u = \cos(\ln(x))\,dx \qquad dv = \ dx$$
$$du = -\sin(\ln(x)) \cdot \tfrac{1}{x}\ dx \qquad v = x$$

$$\begin{aligned}
\int \sin(\ln(x))\,dx &= x\sin(\ln(x)) - \boxed{\left[x\cos(\ln(x)) + \int \sin(\ln(x))\,dx\right]} \\
2\int \sin(\ln(x))\,dx &= x\sin(\ln(x)) - x\cos(\ln(x)) \\
\int \sin(\ln(x))\,dx &= \frac{1}{2}(x\sin(\ln(x)) - x\cos(\ln(x))) + C
\end{aligned}$$

Solutions

I(4) $\int x^3 e^{x^2}\, dx$

- Parts with $u = x^3$ & $dv = e^{x^2}$ doesn't work- can't do $\int e^{x^2}\, dx$.
- Parts with $u = e^{x^2}$ & $dv = x^3$ doesn't help - end up with $\int x^5 e^{x^2}\, dx$
- Notice that $x^3 = x^2 \cdot x$

$u = x^2$ $\qquad dv = xe^{x^2}\, dx$

$du = 2x\, dx$ $\qquad$ To find v, use substitution!

Let $w = x^2$. Then $dw = 2x\, dx$, and so $\frac{1}{2}\, dw = x\, dx$.

Therefore $v = \int xe^{x^2}\, dx = \int \frac{1}{2} e^w\, dw = \frac{1}{2} e^w = \frac{1}{2} e^{x^2}$.

$$\begin{aligned}\int x^3 e^{x^2}\, dx &= \frac{x^2}{2} e^{x^2} - \frac{1}{2} \int e^{x^2} \cdot 2x\, dx = \frac{x^2}{2} e^{x^2} - \int xe^{x^2}\, dx \\ &= \frac{x^2}{2} e^{x^2} - \frac{1}{2} e^{x^2} + C\end{aligned}$$

Pointers

II(1) $\displaystyle\int_1^{e^3} \frac{\ln(x)}{x}\,dx$

Substitution: If we write this as a product, $\ln(x) \cdot \frac{1}{x}$, we can see that the two pieces are related.
Let $u = \ln(x)$, so $du = \frac{1}{x}\,dx$.

II(2) $\displaystyle\int \sin^2(x)\,dx$

Integration by Parts: Letting $u = \sin(x)$ *and* $dv = \sin(x)\,dx$ leads to something involving $\displaystyle\int \cos^2(x)\,dx$.

Using parts again doesn't end up working (you end up getting $\displaystyle\int \sin^2(x)\,dx = \int \sin^2(x)\,dx$, which is true but not very informative.

Instead, now use the trig identity $\cos^2(x) = 1 - \sin^2(x)$.

Pointers

II(3) $\int x^3 \cos(x^4)\, dx$.

Substitution: We have a composition, $\cos(x^4)$. Furthermore, we more or less have the derivative of the inside function.
Let $u = x^4$, so $du = 4x^3\, dx$.

II(4) $\int \cos(x^{1/3})\, dx$

Parts and substitution: We have composition, so begin with
$u = x^{1/3}$, so $du = \frac{1}{3}x^{-2/3}\, dx \Rightarrow 3x^{2/3}\, du = dx \Rightarrow 3u^2\, du = dx$.
Make the substitution, use integration by parts a time or two.

Solutions

II(5) $\int \arctan(x)\, dx$

Let

$$u = \arctan(x) \qquad dv = dx$$
$$du = \frac{1}{1+x^2}\, dx \qquad v = x$$

Thus

$$\begin{aligned}\int \arctan(x)\, dx &= x\arctan(x) - \int \frac{x}{1+x^2}\, dx \\ &= x\arctan(x) - \frac{1}{2}\ln(1+x^2) + C\end{aligned}$$

Verify:

$$\begin{aligned}\frac{d}{dx}\left(x\arctan(x) - \frac{1}{2}\ln(1+x^2) + C\right) &= \frac{x}{1+x^2} + \arctan(x) - \frac{1}{2}\frac{1}{1+x^2}(2x \\ &= \arctan(x).\end{aligned}$$