WeBWorK Problem 4

Evaluate $\displaystyle\int x \arctan(3x)\ dx$

$u = \arctan(3x)$	$dv = x\ dx$
$\Downarrow$	$\Downarrow$
$du = \dfrac{1}{1+(3x)^2} \cdot 3$	$v = \dfrac{1}{2}x^2$

$$\begin{aligned} \int x \arctan(3x)\ dx &= uv - \int v\ du \\ &= \frac{x^2}{2}\arctan(3x) - \frac{3}{2}\int \frac{x^2}{1+9x^2}\ dx \end{aligned}$$

How do we find $\displaystyle\int \frac{x^2}{1+9x^2}\ dx$?

WeBWorK Problem 4, continued

Method #1 - the way I showed in class. For an easier way, see next slide

$$\int x \arctan(3x)\,dx = \frac{x^2}{2}\arctan(3x) - \frac{3}{2}\int \frac{x^2}{1+9x^2}\,dx$$

$$\begin{aligned}
\int \frac{x^2}{1+9x^2}\,dx &= \int \frac{\overbrace{\frac{1}{9}\cdot 9}^{1}\,x^2}{1+9x^2}\,dx = \frac{1}{9}\int \frac{9x^2 + \overbrace{1-1}^{0}}{1+9x^2}\,dx \\
&= \frac{1}{9}\int \frac{1+9x^2}{1+9x^2} - \frac{1}{1+9x^2}\,dx = \frac{1}{9}\int 1 - \frac{1}{1+9x^2}\,dx \\
&= \frac{1}{9}\int 1\,dx - \frac{1}{9}\cdot\frac{1}{3}\int \frac{3}{1+(3x)^2}\,dx = \frac{x}{9} - \frac{\arctan(3x)}{27} + C
\end{aligned}$$

$$\int x \arctan(3x)\,dx = \frac{x^2}{2}\arctan(3x) - \frac{3}{2}\left(\frac{x}{9} - \frac{\arctan(3x)}{27}\right) + C$$

WeBWorK Problem 4, continued

Method #2 - as suggested by a student

$$\int x \arctan(3x)\, dx = \frac{x^2}{2}\arctan(3x) - \frac{3}{2}\int \frac{x^2}{1+9x^2}\, dx$$

Use polynomial long division to rewrite $\dfrac{x^2}{9x^2+1}$

$$\begin{array}{r|l} & \frac{1}{9} \\ \hline 9x^2+1 & x^2 \\ & -x^2 - \frac{1}{9} \\ \hline & -\frac{1}{9} \end{array}$$

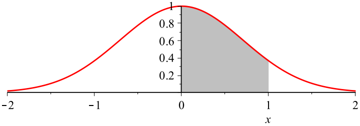
So we can rewrite $\dfrac{x^2}{1+9x^2}$ as $\dfrac{1}{9} - \dfrac{\frac{1}{9}}{1+9x^2}$

Thus $\displaystyle\int \frac{x^2}{1+9x^2}\, dx = \frac{1}{9}\int 1 - \frac{1}{1+9x^2}\, dx$, which is what we got with the first method, but without any "tricks".

Recall:

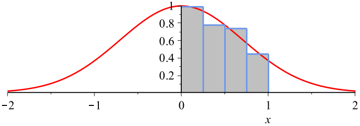
- We originally defined the definite integral to the signed area between a function and the x-axis.

$\int_0^1 e^{-x^2}\,dx$ is the shaded area



- We can approximate this signed area by using rectangles whose height is the value of the function somewhere over the sub-interval the rectangle sits in.

Approximate $\int_0^1 e^{-x^2}\,dx$ with rectangles

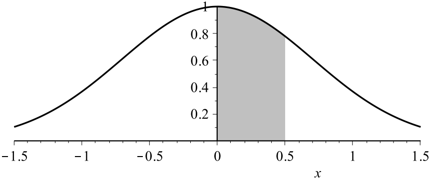


In Class Work - Sigma Notation:

1. Write $2^2 + 3^2 + 4^2 + 5^2 + 6^2$ in sigma notation, starting with $i = 2$.
2. Again write $2^2 + 3^2 + 4^2 + 5^2 + 6^2$ in sigma notation, but this time start with $i = 0$. How does the term you're adding change? How high will i go?
3. Write $\frac{\pi}{4}\sin\left(\frac{\pi}{4}\right) + \frac{\pi}{4}\sin\left(\frac{\pi}{2}\right) + \frac{\pi}{4}\sin\left(\frac{3\pi}{4}\right) + \frac{\pi}{4}\sin(\pi)$ in sigma notation. This is R_4 for $\int_0^\pi \sin(x)\,dx$.
4. Write $\frac{\pi}{4}\sin(0) + \frac{\pi}{4}\sin\left(\frac{\pi}{4}\right) + \frac{\pi}{4}\sin\left(\frac{\pi}{2}\right) + \frac{\pi}{4}\sin\left(\frac{3\pi}{4}\right)$ in sigma notation. This is L_4 for $\int_0^\pi \sin(x)\,dx$.

In Class Work - Approximating Sums

Let $\mathcal{I} = \int_0^{1/2} e^{-x^2}\,dx$.



1. Sketch R_4, the right-sum approximation for $\mathcal{I}$. Will R_4 over- or under-estimate $\mathcal{I}$?
2. By hand, write R_4 as a sum (w/out sigma notation); calculate.
3. Repeat Problems 1 and 2 for L_4, M_4, and T_4. (You may use that T_4 is the average of L_4 and R_4.)
4. Use your answer to Problem 2 to write R_4 in sigma notation. (Look for patterns, don't just use a formula.)

Solutions: Sigma Notation:

1. Write $2^2 + 3^2 + 4^2 + 5^2 + 6^2$ in sigma notation, starting with $i = 2$.
   - Adding up consecutive squares: every term has the form i^2.
   - First term is 2^2 & last is 6^2, so start with $i = 2$ and end with $i = 6$.
   - $$2^2 + 3^2 + 4^2 + 5^2 + 6^2 = \sum_{i=2}^{6} i^2.$$
2. Repeat, but start with $i = 0$.
   - $2^2 \leftrightarrow i = 0$, $3^2 \leftrightarrow i = 1$, $4^2 \leftrightarrow i = 2$, $5^2 \leftrightarrow i = 3$, $6^2 \leftrightarrow i = 4$.
   - Thus if i begins at 0 it ends at 4.
   - Also notice: every term we're squaring is always $i + 2$, so terms are $(i + 2)^2$.
   - $$2^2 + 3^2 + 4^2 + 5^2 + 6^2 = \sum_{i=0}^{4} (i + 2)^2.$$

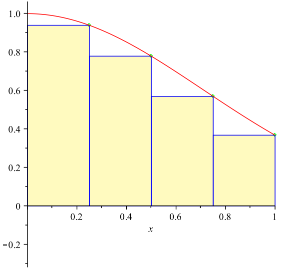
Solutions: Sigma Notation:

3. Write $\frac{\pi}{4}\sin\left(\frac{\pi}{4}\right)+\frac{\pi}{4}\sin\left(\frac{\pi}{2}\right)+\frac{\pi}{4}\sin\left(\frac{3\pi}{4}\right)+\frac{\pi}{4}\sin(\pi)$ in sigma notation.
   - Every term has the form $\frac{\pi}{4}\sin\left(\frac{i\pi}{4}\right)$, with i going from 1 to 4.
   - $$\frac{\pi}{4}\sin\left(\frac{\pi}{4}\right)+\frac{\pi}{4}\sin\left(\frac{\pi}{2}\right)+\frac{\pi}{4}\sin\left(\frac{3\pi}{4}\right)+\frac{\pi}{4}\sin(\pi)=\sum_{i=1}^{4}\frac{\pi}{4}\sin\left(\frac{i\pi}{4}\right).$$
4. Write $\frac{\pi}{4}\sin(0)+\frac{\pi}{4}\sin\left(\frac{\pi}{4}\right)+\frac{\pi}{4}\sin\left(\frac{\pi}{2}\right)+\frac{\pi}{4}\sin\left(\frac{3\pi}{4}\right)$ in sigma notation.
   - Every term has the form $\frac{\pi}{4}\sin\left(\frac{i\pi}{4}\right)$, with i going from 0 to 3.
   - $$\frac{\pi}{4}\sin(0)+\frac{\pi}{4}\sin\left(\frac{\pi}{4}\right)+\frac{\pi}{4}\sin\left(\frac{\pi}{2}\right)+\frac{\pi}{4}\sin\left(\frac{3\pi}{4}\right)=\sum_{i=0}^{3}\frac{\pi}{4}\sin\left(\frac{i\pi}{4}\right).$$

Solutions: Approximating Sums

For $\mathcal{I} = \int_0^1 e^{-x^2}\,dx$:

3. By hand, sketch R_4, the right-sum approximation for $\mathcal{I}$.



Because f is decreasing, R_4 is an under-estimate

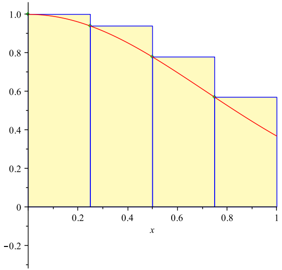
4. By hand, write R_4 as a sum (without sigma notation), and then calculate.

$$\begin{aligned} R_4 &= \frac{1}{4}\cdot e^{-(1/4)^2} + \frac{1}{4}\cdot e^{-(1/2)^2} \\ &\quad + \frac{1}{4}\cdot e^{-(3/4)^2} + \frac{1}{4}\cdot e^{-(1)^2} \\ &= \frac{1}{4}\Big(e^{-1/16} + e^{-1/4} + e^{-9/16} \\ &\quad + e^{-1}\Big) \\ &\approx 0.66396 \end{aligned}$$

Solutions: Approximating Sums

For $\mathcal{I} = \int_0^1 e^{-x^2}\, dx$:

5 (a). Repeat Problems 3 and 4 for L_4.



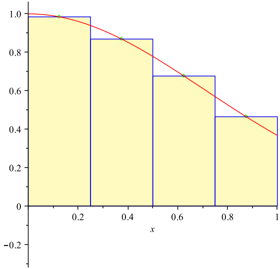
Because f is decreasing, L_4 is an over-estimate

$$
\begin{aligned}
L_4 &= \frac{1}{4} \cdot e^{-0^2} + \frac{1}{4} \cdot e^{-(1/4)^2} \\
&\quad + \frac{1}{4} \cdot e^{-(1/2)^2} + \frac{1}{4} \cdot e^{-(3/4)^2} \\
&= \frac{1}{4}\Big(1 + e^{-1/16} + e^{-1/4} \\
&\quad + e^{-9/16}\Big) \\
&\approx 0.82199
\end{aligned}
$$

Solutions: Approximating Sums

For $\mathcal{I} = \int_0^1 e^{-x^2}\,dx$:

5 (b) Repeat Problems 3 and 4 for M_4.



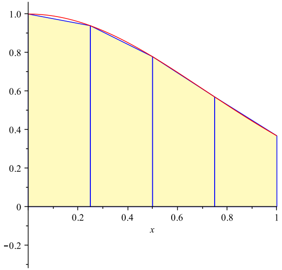
Hard to tell; turns out that because f is concave down, M_4 is an over-estimate

$$\begin{aligned} M_4 &= \frac{1}{4}\cdot e^{-(1/8)^2} + \frac{1}{4}\cdot e^{-(3/8)^2} \\ &\quad + \frac{1}{4}\cdot e^{-(5/8)^2} + \frac{1}{4}\cdot e^{-(7/8)^2} \\ &= \frac{1}{4}\Big(e^{-1/64} + e^{-9/64} \\ &\qquad + e^{-25/64} + e^{-49/64} \Big) \\ &\approx 0.74874 \end{aligned}$$

Solutions: Approximating Sums

For $\mathcal{I} = \int_0^1 e^{-x^2}\,dx$:

5 (c) Repeat Problems 3 and 4 for T_4. (You may want to use that T_4 is the average of L_4 and R_4.)



$$
\begin{aligned}
T_4 &= \frac{L_4 + R_4}{2} \\
&\approx \frac{0.66396 + 0.82199}{2} \\
&\approx 0.74298
\end{aligned}
$$

Because f is concave down, T_4 is an under-estimate

Solutions: Approximating Sums

For $\mathcal{I} = \int_0^1 e^{-x^2}\, dx$:

6. Use your answer to Problem 4 to write R_4 in sigma notation. (Look at Problem 4 for patterns, don't just look in the book for a formula.)

$$
\begin{aligned}
R_4 &= \frac{1}{4}\cdot e^{-(1/4)^2} + \frac{1}{4}\cdot e^{-(1/2)^2} + \frac{1}{4}\cdot e^{-(3/4)^2} + \frac{1}{4}\cdot e^{-(1)^2} \\
&= \sum_{i=1}^{4} \frac{1}{4} e^{-(i/4)^2} \\
&= \frac{1}{4}\sum_{i=1}^{4} e^{-i^2/16}
\end{aligned}
$$