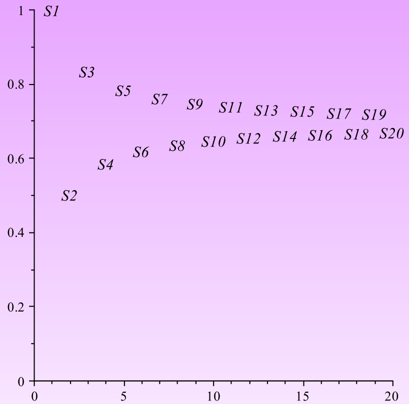
Recall:

Let $\sum_{k=1}^{\infty} c_k$ be any series. (The c_k may be positive or negative).

- If $\sum |c_k|$ diverges but $\sum c_k$ converges, then $\sum c_k$ converges **conditionally**.
- If $\sum |c_k|$ and $\sum c_k$ both converge, then we say $\sum c_k$ converges **absolutely**.
  - **Note:** If $\sum |c_k|$ converges, then $\sum c_k$ converges as well, and so $\sum c_k$ converges **absolutely**.

Alternating Harmonic Series, $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$



We can see the Alternating Harmonic Series converges to some value between $\frac{1}{2}$ and 1.

Rearranging Terms so the AHS converges to 1.5:

1. Add up enough positive terms to go *just* over 1.5, in order of their appearance
$$1 + \frac{1}{3} + \frac{1}{5} = 1.5333...$$
2. Add in enough negative terms to go *just* below 1.5 (in order of their appearance)
$$1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{2} = 1.0333...$$
3. Now include the next positive terms, until I just go over 1.5 again.
$$1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{2} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \frac{1}{15} = 1.521800422...$$
4. Again include enough negative terms (in order) to go just below 1.5
$$1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{2} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \frac{1}{15} - \frac{1}{4} = 1.271800422...$$

Rearranging Terms so the AHS converges to 1.5:

Here are the first 20 partial sums created by adding in this order:

$$
\begin{array}{rcl}
1.533333333 & \Rightarrow & 1.033333333 \\
\Rightarrow 1.521800422 & \Rightarrow & 1.271800422 \\
\Rightarrow 1.514352839 & \Rightarrow & 1.347686172 \\
\Rightarrow 1.510338491 & \Rightarrow & 1.385338491 \\
\Rightarrow 1.507874824 & \Rightarrow & 1.407874824 \\
\Rightarrow 1.506217169 & \Rightarrow & 1.422883836 \\
\Rightarrow 1.505027922 & \Rightarrow & 1.433599351 \\
\Rightarrow 1.504133948 & \Rightarrow & 1.441633948 \\
\Rightarrow 1.503437767 & \Rightarrow & 1.447882211 \\
\Rightarrow 1.502880434 & \Rightarrow & 1.452880434
\end{array}
$$

In Class Work

Use the ratio test to decide whether the following alternating series convergence absolutely, converge conditionally, or diverge:

1. $\displaystyle\sum_{k=12}^{\infty} \frac{(-10)^k}{k!}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{n^{50}}$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{n!}$

Solutions

1. $\displaystyle\sum_{k=12}^{\infty} \frac{(-10)^k}{k!}$

Since none of the terms are ever zero, the ratio test applies.

$$L = \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \left| \frac{(-10)^{k+1}}{(k+1)!} \cdot \frac{k!}{(-10)^k} \right| = \lim_{k\to\infty} \frac{10}{k+1} = 0.$$

Since $L < 1$, the ratio test tells us this series **converges**!

Solutions

2. $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^{50}}$

Again, for all $n \geq 1$, the terms are non-zero, so the ratio test applies. Whether or not it's conclusive remains to be seen.

Let

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-2)^{n+1}}{(n+1)^{50}} \cdot \frac{n^{50}}{(-2)^n} \right| = \lim_{n\to\infty} 2 \left(\frac{n}{n+1} \right)^{50} = 2.$$

Since $L > 1$, the ratio test tells us that this series **diverges.**

Solutions

3. $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{n!}$

None of the terms are ever 0, so the ratio test applies.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)^3}{(n+1)!}\cdot\frac{n!}{n^3} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^3\frac{1}{n+1} = 1\cdot 0 = 0 < 1$$

Since $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$, the series $\displaystyle\sum_{n=0}^{\infty}\frac{n^3}{n!}$ converges, by the ratio test.