In Class Work

Let $P(x)$ be the power series

$$1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \cdots = \sum_{k=0}^{\infty} \frac{(-x)^k}{k+1}$$

1. Does the series converge when $x = 1$?
2. Does the series converge when $x = -1$?
3. Does the series converge when $x = \frac{1}{2}$?
4. For what values of x does $P(x)$ converge absolutely? (Remember: Try the Ratio Test)

Solutions:

Let $P(x)$ be the power series

$$1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \cdots = \sum_{k=0}^{\infty} \frac{(-x)^k}{k+1}$$

1. Does the series converge when $x = 1$?

   When $x = 1$, we have

   $$P(1) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1}.$$

   This is an alternating series, and the alternating series test immediately tells us the series $P(1)$ converges.

Solutions:

2. Does the series converge when $x = -1$?

   When $x = -1$, we have

   $$P(-1) = \sum_{k=0}^{\infty} \frac{(1)^k}{k+1}.$$

   Positive-term series: use the Integral Test (if it applies)

   $a(x) = 1/(x+1)$ is $+$, cont., $\downarrow$ on $[0, \infty)$, so Int Test applies

   $\displaystyle\int_0^{\infty} \frac{1}{x+1}\,dx$ diverges, and so we know the series $P(-1)$ does also.

   (By the way, this also tells us that the first series we looked at, $P(1)$ only converges conditionally, not absolutely.)

Solutions:

3. Does the series converge when $x = \frac{1}{2}$?

$$P(1/2) = \sum_{k=0}^{\infty} \frac{(-1/2)^k}{k+1} = \sum_{k=0}^{\infty} (-1)^k \frac{1}{2^k(k+1)}.$$

By the alternating series test, this series converges.

In fact, it converges absolutely, which we can conclude by comparing $\displaystyle\sum_{k=0}^{\infty} \frac{1}{2^k(k+1)}$ to the larger convergent series $\displaystyle\sum_{k=0}^{\infty} \frac{1}{2^k}$.

Solutions:

4. For what values of x does $P(x)$ converge absolutely? (Ratio Test)

$$\begin{aligned} L &= \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \frac{|x|^{k+1}}{k+2} \cdot \frac{k+1}{|x|^k} \\ &= \lim_{k\to\infty} |x| \cdot \frac{k+1}{k+2} = |x| \end{aligned}$$

Therefore the series converges absolutely whenever $|x| < 1$, that is, when $-1 < x < 1$, and diverges when $|x| > 1$.

Although the Ratio Test is inconclusive when $x = \pm 1$, we already figured out what the series does for those two possible values of x.

Thus $\displaystyle\sum_{k=0}^{\infty} \frac{(-x)^k}{k+1}$ converges on the interval $-1 < x \leq 1$, converging absolutely on the interior and conditionally at $x = 1$.